Universality in Random Matrix Models of Quantum Chromodynamics

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Abstract of the Dissertation

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In this thesis we investigate various spectral properties of random matrix models of quantum chromodynamics (QCD) and find that some of them are *universal* in the sense that their functional form is invariant under various deformations of the random matrix ensemble in question. In particular the microscopic part of the spectral density is such a universal distribution, which was shown to coincide with lattice QCD data. This is of physical importance because the microscopic part of the Dirac spectrum carries information about the chiral symmetry breaking in QCD. We also develop analytical tools for similar studies, including Harish-

Chandra-Itzykson-Zuber type integrals relevant for calculating ensemble averages in chiral random matrix ensembles and a method to connect the spectral information of a chiral unitary ensemble to the corresponding orthogonal and symplectic ensembles.

to Tijen

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Chapter 1

Introduction

Except for textbook examples, problems in physics rarely lend themselves for exact solution. Therefore a physicist is often led to work with approximation schemes or *models*. A model may disregard certain details of the system in question and therefore may work - if at all - in one domain, but remain a bad approximation in another. Furthermore, it is usually the case that, given the circumstances, one property of a physical system is more important than others and a full description of the system is not needed for answering the questions about that property. Hence one desires to have some means of deducing the consequences and the relevance of each individual bit of information that goes into the definition of a given physical system. In certain circumstances random matrix models provide partial solutions to these problems.

In the construction of a random matrix model for a given physical system one starts with identifying a linear operator which carries information about the dynamics of the system. For example this would usually be the hamiltonian for a quantum mechanical system. Typically one desires to learn about its spectrum which may defy an analytical description.

Random matrix theory proceeds by identifying fundamental symmetries and constraints about the structure of this linear operator. For example, it may be complex hermitian or real symmetric or it might posses a certain block structure, it might have commutativity properties with other operators, *et cetera* . Once important symmetries and constraints are identified, all other system specific information is 'erased' from the model. Discarding of information essentially corresponds to taking irrelevant details to be random. Thus the actual linear operator which contains full dynamical information is replaced by an ensemble of operators (matrices) which are random except for the decided constraints and symmetries imposed on them. In most cases an average over such an ensemble is much easier to evaluate than computing the spectrum for the full problem.

'Conservation of difficulty' would lead one to expect that this kind of an approach is bound to fail for all but a few questions asked about the physical system in question. As most system specific information is discarded in the construction of the random matrix model, any properties dependent on them can no longer be recovered. The only things that can be recovered are the *universal* properties. By a universal property we mean those properties that are shared by almost all systems belonging to the ensemble, as defined by the given symmetry and constraints. (Probably a better way to define universality is as 'invariance under deformations' of the original problem as we will see later on.) In other words, computation of universal properties usually do not require an analytical solution of the full physical problem. The corresponding random matrix model might provide the same answer more easily.

Historically, random matrix theory was invented by Wigner in the 1950's [223, 224] in an attempt to gain a statistical understanding of nuclear spectra. Wigner observed that it was practically impossible to compute analytically the positions of the thousands of energy levels observed for a given atomic nucleus. The hamiltonian, which carries all dynamical information about the nucleus, was too complicated for an exact study, and Wigner was therefore led to a statistical study of the energy levels.

Wigner decided to examine the spectrum of a random hamiltonian, described by independently drawn gaussian random numbers as matrix elements except for the constraint of being real symmetric. The overall average spectral density of such a model is a semicircular distribution centered around zero; however, the spectral density for most physical systems is a monotonically increasing function. Thus it was clear that such an approach cannot tell much about the overall shape of the spectrum. Wigner realized, however, that a random matrix model does a good job predicting *local* spectral statistics, whereby local we mean at the scale of the mean energy level spacing. In fact the distribution of the distance between neighboring energy levels (after some 'unfolding' procedure as explained in chapter 2) is precisely the same as that given by the random matrix model. This agreement provided the initial impetus for the development of random matrix theory. More details will be discussed later in chapter 2.

An interesting perspective on random matrix theories is to consider them as *'minimal information models'*. This view was established by Balian [18] in the 1960's. He realized that some canonical random matrix ensembles can be reconstructed by a procedure where one treats the probability distribution defining the ensemble as a variable and minimizes the information content of the probability distribution for the ensemble, modulo the constraints and

symmetries imposed on the ensemble. Minimizing the amount of information is formally the same as maximizing the entropy, or the randomness, of the remaining (unconstrained) properties of the system. As such, Balian's framework provides a means to study the consequences of individual pieces of information separated from other system specific details. We will discuss Balian's method in more detail in chapter 2.

The reason why random matrix theory works at all is because of *universality* encountered in certain classes of dynamical systems. As universality (at least in the present context) is not an entirely well defined concept; it is better to proceed by example, namely, by the most well known case of universality in quantum mechanical spectra of classically chaotic dynamical systems. Bohigas, Giannoni and Schmidt [36] conjectured in 1984 that local spectral properties of quantum mechanical systems whose classical analogs are chaotic (and are time-reversal invariant), are identical to those given by the so called gaussian orthogonal random matrix ensemble. They have reached their statement via a number of numerical studies concerning such spectra. Their conjecture has been extensively tested on many examples to this day.

The Bohigas-Giannoni-Schmidt conjecture essentially states that the aforementioned spectral properties are system independent, up to the point of changing the fundamental symmetries of the system. (For example turning on a magnetic field which might break the time-reversal symmetry, will change the 'universality class' in question.) As these properties are shared by almost all systems obeying the same constraints they can therefore be computed in any one of them, preferably a simpler one.

Within the context of random matrix theory, the phrase 'universality' takes a more formal meaning. The definition of a random matrix ensemble not only requires the choice of a subset among matrices but also to assign a weight function with respect to which all averages should be performed. One typically works with gaussian distributions as they are easier to handle; however for the outcomes of the model to have any significance the arbitrary choice of a weight function should not have any role. In fact, as we will discuss in chapter 4, almost all significant properties of gaussian ensembles (by which we mean those properties that tend to agree with those of corresponding real physical systems) are invariant under various deformations of the defining weight function.

This form of 'universality' is reminiscent of the central limit theorem. The central limit theorem establishes a mathematical ground for the ubiquity of the appearance of the bell curve. Many complex interacting systems (may they be of social, biological or physical origin) show the same behavior regarding the distribution of certain magnitudes (test scores, heights of people, deviations

in a measurement etc.). Although a detailed understanding of most of these systems is out of reach, the ubiquity of the gaussian distribution is easier to understand. If one starts with a probability distribution and defines a new distribution by the sum of two random numbers drawn from the initial probability distribution, repeating this procedure will eventually converge on the gaussian distribution for most starting distributions. In other words, the gaussian distribution is an attracting fixed point of a renormalization group flow. (In chapter 2 we will outline a proof of a similar statement as given by Feinberg and Zee.)

Like the central limit theorem, universality studies in the context of random matrix theory establish certain spectral distributions to be invariant under changes of the defining probability distribution for the ensemble in question. However, unlike in the case of the central limit theorem, general proofs are harder to come by and usually each variation of a model requires separate study. The work to be outlined below has consisted of contributions in this direction. It is mostly analytical in nature, where numerical studies were usually used only at the experimenting stage for testing new ideas.

We study a rather specific class of random matrix theories below, namely those inspired by quantum chromodynamics. Quantum chromodynamics (QCD) is believed to be a viable microscopic theory describing nuclear interactions. Although it can be described rather easily in terms of a local lagrangian, a direct deduction of many of its consequences have remained elusive so far. This has led to many QCD inspired models as well as various numerical studies. As will be discussed in detail below, certain physical questions regarding QCD can be formulated in terms of the spectrum of the Dirac operator, which is the linear operator carrying the relevant information in this case. The scope of the studies of QCD inspired random matrix models is to establish certain characteristics of the Dirac spectra as universal and so derive them from a corresponding random matrix model. It is important to emphasize at this point that a direct computation of such spectral data from fundamentals of QCD is generally out of reach and the alternative typically is the numerical studies done on a discretized lattice of space-time. Random matrix theories naturally complement such numerical studies.

Today random matrix theory enjoys a renewed interest after a certain period of being dormant before mid 1980's. It is probably fair to say that universality studies to this day provide some form of a 'tool box' which must be refined for each individual problem encountered, rather than an all encompassing general theory. Random matrix theories have been applied to many branches in and out of physics including quantum mechanical spectra, the spectrum of the Dirac operator in quantum chromodynamics, 2D gravity,

condensed matter physics, resonance spectra in microwave cavities, elastomechanics, neural networks and the spectrum of the zeroes of the Riemann zeta function, to name a few. Various references can be found throughout the following chapters and the bibliography of this thesis.

The outline of this thesis is as follows: In chapter 2 we will introduce random matrix theories on a more formal basis from a partially historical perspective and establish some of the basic relations used throughout this work. Then in chapter 3 we will briefly summarize some of the relevant properties of quantum chromodynamics with an emphasis on chiral symmetry breaking. Chapter 4 is the core of this text and will mainly consist of published work [125, 124, 126, 177] along the lines outlined before. Various computational conventions as well as brief introductory treatments of some of the analytical techniques used throughout are relegated to the appendices.

Chapter 2

Random matrix theories: An Introduction

In this chapter we give a brief introduction on general concepts of random matrix theory. As a complete treatment of this subject is beyond the scope of this manuscript we will naturally be biased in the selection of the topics. Priority will be given to those that are most relevant for this work. For a more complete treatment of random matrix theory the reader is referred to some standard references. Especially the treatise by Mehta [149] and the early preprint collection by Porter [167] serve this purpose more than adequately. Furthermore a relatively recent review of common concepts in random matrix theory has been published by Guhr, Müller-Groeling and Weidenmüller [101], which has a more modern perspective than the former two references.

We will start with a historical survey of random matrix theory in section 2.1, where some points already mentioned in the introduction will be discussed in more detail. We will try to maintain the continuity at the cost of some redundancy. Then we will establish some useful concepts regarding the statistics of the eigenvalues of a random matrix in section 2.2. Sections 2.3 and 2.4 introduce two of the commonly used techniques in random matrix theory, namely the orthogonal polynomial and the supersymmetry methods. In section 2.5 we will introduce the information theoretical approach of Balian to random matrix theory. This will be followed by a discussion of universality in the context of random matrix theory and a (rather informal) analogy to the central limit theorem in section 2.6.

2.1 A historical perspective on Random Matrix Theory

Random matrix theory was introduced in physics by Wigner in 1951 [223, 224] in an attempt to formulate a statistical understanding of the resonances

observed in the scattering of slow neutrons across nuclei. This was motivated by the measurement of sufficiently long level sequences with enough resolution to make statistical studies of the spectrum feasible.

The aforementioned resonances correspond to long lived compound states formed by the incoming neutron and the target nucleus [101, 38]. At that time no generally accepted dynamical theory for their study was available. Furthermore, for a system with many degrees of freedom and strong interactions, a detailed study of the resonances was elusive. Since there was no means of computing the energy levels of a system as complex as a nucleus far away from the ground state, Wigner considered the possibility of investigating these levels statistically.

A statistical study of energy levels is a rather unorthodox approach - especially for the time frame mentioned above. A hamiltonian, though it may be unknown or complicated, *deterministically* specifies the energy levels of a system. There is no uncertainty involved. However, if one is not concerned with the exact positions of energy levels one might hope to understand the relations between levels within a certain interval such as correlations or even perhaps to gain an understanding of the average spectral density. Thus Wigner considered the spectrum of a random matrix which shares the symmetries of the given Hamiltonian. (For example a real symmetric matrix for a hamiltonian invariant under time reversal symmetry as we will see below.)

For a real symmetric matrix with entries drawn independently from a gaussian distribution the spectral properties are well known [149]. The average spectral density is a semi-circle centered around zero. However, for many physical systems the average spectral density is a monotonically increasing function of the energy. Therefore it is clear that such an approach cannot say much about the overall shape of the spectral density - the details of the system ignored by the random matrix model are important for the determination of the average spectral density. Fortunately more can be said about local properties of such spectra.

By local properties of the spectrum, we mean those properties that are relevant at the scale of the mean eigenvalue spacing. The nearest neighbor spacing distribution is one such property. It is the distribution of the spacings between neighboring eigenvalues in an 'unfolded' spectrum. (Unfolding is a procedure which maps the spectrum under study to another one with a fixed mean local eigenvalue spacing while respecting relative positions of the eigenvalues - i.e. global information is sacrificed for a more convenient study of the local characteristics. We will see details of this procedure below.)

For the nearest neighbor spacing distribution of nuclear levels Wigner proposed two rules: [149, 226] 1) Levels with different spin and parity are un-

correlated. 2) For levels with the same spin and parity the spacing distribution is given by the now so called *Wigner surmise*:

$$p(s) = \frac{\pi s}{2} e^{-\frac{\pi}{4}s^2}, \tag{2.1}$$

where s denotes the spacing between two neighboring levels as measured in terms of the local mean level spacing. (The scale is chosen such that the average value of s in (2.1) is unity.)

The Wigner surmise follows from two relatively simple assumptions [149], namely, that for small s the probability of finding a level at $E+s$ is proportional to s given that there is a level at E and that the probabilities in various subintervals of $(E, E+s)$ are mutually independent. As Mehta points out this corresponds to the assumption of vanishing higher correlation functions. Hence we get

$$\begin{aligned} p(s)ds &= b \lim_{m\to\infty} \prod_{r=0}^{m-1} \left(1 - \frac{sr}{m}\frac{s}{m}a\right) asds \\ &= abse^{-as^2/2}ds, \end{aligned} \tag{2.2}$$

where a and b are fixed by the average value of s and the normalization.

Amazingly (2.1) coincides with the distribution of the spacing between the two eigenvalues of a 2×2 random matrix and is remarkably close (but not identical) to the spacing distribution for an arbitrarily large random matrix. As it will be discussed below, this distribution (that of a large matrix) precisely agrees with the spacing distribution of certain complex quantum mechanical systems. The asymptotic form of the spacing distribution for large matrices can be computed analytically, but the computation is rather involved. We refer the reader to [149] instead.

As the example above illustrates, random matrix models are successful usually at the scales of the mean eigenvalue spacing and global characteristics of the spectrum are beyond their prediction range. (However using Balian's method [18] introduced below, one can 'tune' a random matrix model to have a given average spectral density.)

Dyson, called random matrix theory as 'a new kind of statistical mechanics'. In statistical mechanics one typically considers a collection of identical dynamical systems realized at different instances in the phase space, the averages therefore are over those instances (with respect to the Boltzmann distribution). Another kind of averaging comes up in number theory, where for a fixed distribution of prime numbers one works with a running average over the 'spectrum'. Contrasting the former two examples, in random matrix

theory the averaging is done over an *ensemble* of different dynamical systems (as represented by different linear operators) sharing similar symmetry and constraints. Ensemble averages are performed as a weighted sum over this set of dynamical systems. This kind of averaging wipes out any system specific information. Only those properties that are shared by almost all dynamical systems in the ensemble survive the averaging procedure: We refer to these properties as *universal properties*. There is no generally accepted formal definition of universality in random matrix theory as of now. However it is generally agreed that, to call a property universal it should not depend on the details of the choice of the weight function imposed on the ensemble. This will be illustrated by many examples in chapter 4.

Wigner's early work was followed by Dyson, Mehta, Porter, Thomas, Kahn and others [72, 150, 68, 68, 69, 70, 71, 147, 151, 134, 86, 168]. In particular, Dyson came up with a classification of generic ensembles in the framework of nonrelativistic quantum mechanics, which is now called *the three-fold way* [67]. According to this classification, time reversal invariant systems with rotational symmetry (and time reversal invariant systems with integer spin and broken rotational symmetry) are described by a real symmetric hamiltonian:

$$H = H^T = H^*. \tag{2.3}$$

Systems that are *not* invariant under time reversal symmetry (such as a charged particle in an external magnetic field) are described by a complex hermitian hamiltonian:

$$H = H^\dagger \tag{2.4}$$

Finally, systems with half integer spin and time reversal invariance are described by self-dual quaternion matrices:

$$H = e_2 H^T e_2, \tag{2.5}$$

where H consists of 2×2 blocks representing quaternions in the form

$$q = q_0 \mathbf{1}_2 + q_1 e_1 + q_2 e_2 + q_3 e_3. \tag{2.6}$$

Here the generators of the quaternion algebra are represented in the matrix form

$$e_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \; e_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \; e_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}; \tag{2.7}$$

$\mathbf{1}_N$ is the $N \times N$ identity matrix and e_2 in (2.5) is meant as $e_2 \otimes \mathbf{1}$. The proof of this classification statement can be found in [149].

The standard random matrix ensembles corresponding to those three conditions (sometimes called Dyson ensembles or canonical ensembles) are described by partition functions defined over all matrices satisfying one of three conditions mentioned above with gaussian weight functions for the distribution of the matrix elements:

$$Z_{RMT} = \int d[H] \exp\left(-\alpha \mathrm{Tr}\, H^2\right), \tag{2.8}$$

where $\int d[H]$ is an integration over all independent degrees of freedom of H.

These random matrix models are called the gaussian orthogonal ensemble, the gaussian unitary ensemble and the gaussian symplectic ensemble, respectively, and are identified by the *Dyson index*, $\beta = 1, 2$ or 4, measuring the number of degrees of freedom per matrix element. Hence, there is one canonical ensemble corresponding to each of the three division algebras: the real numbers, the complex numbers and the quaternions. (One naturally wonders about a generalization to octonions for $\beta = 8$, the only remaining division algebra in mathematics; however, it appears that the cost of sacrificing associativity that comes with the introduction of octonions is too great, and such efforts have been unsuccessful. Even some basic definitions and theorems in linear algebra are elusive in an octonionic generalization.)

Remarkably, all three ensembles show very distinct spectral characteristics. As we will see below the gaussian assumption for the weight functions is not a strong requirement for the preservation of these characteristics.

As will become clear, it is very useful to express the partition function (2.8) in terms of the eigenvalues, E_k, of H. Since the partition function is invariant under unitary transformations and the jacobian depends only on the eigenvalues this is possible. For $N \times N$ matrices the partition function is given by

$$Z_{RMT} = \int d[E] \Delta^\beta(E) \exp\left(-\frac{1}{2} N \beta \sum_{k=1}^{N} E_k^2\right). \tag{2.9}$$

We adopt normalization conditions standard in the literature and the Vandermonde determinant, $\Delta(E)$, is given by

$$\begin{aligned} \Delta(E) &= \det[E_k^{l-1}]_{k,l=1\ldots N} \\ &= \prod_{k<l} (E_k - E_l). \end{aligned} \tag{2.10}$$

The Vandermonde determinant, $\Delta^\beta(E)$, in (2.9) is the jacobian from the change of variables from H_{jk} to radial coordinates (i.e. eigenvalues and the diagonalizing unitary matrix). For the computation of this jacobian see [119].

Later, around 1960, Dyson introduced another set of ensembles, called *circular ensembles*. The circular ensemble corresponding to any of the three canonical ensembles consists of unitary matrices obtained by exponentiating the matrices in (2.3), (2.4) or (2.5) and assigning the standard (Haar) measure to the ensembles. This way one works with eigenvalues defined on the unit circle, which is compact, instead of working with eigenvalues defined on the real axis, which is noncompact. This reduces some of the arbitrariness in the choice of the weight functions. In particular, the eigenvalue density is constant for circular ensembles, making the unfolding procedure redundant. The matrices which are elements of circular ensembles can no longer be regarded as 'hamiltonians', instead they should be viewed as unitary S-matrices, probably describing a scattering process.

Another contribution by Dyson was the Coulomb gas picture of the eigenvalues: The weight function in (2.9) can be written as

$$\exp\left(-N\beta\sum_{k=1}^{N}E_k^2+\frac{\beta}{2}\sum_{k<l}\log(E_k-E_l)^2\right). \tag{2.11}$$

Thermodynamically this is the free energy of a static one dimensional Coulomb gas of N particles, with positions, E_k, confined by a harmonic oscillator potential held at a temperature $T=1/\beta$. This perspective allows one to more easily understand the overall shape of the spectral density and the local repulsion between eigenvalues characteristic of random matrix theory as seen in the Wigner surmise, (2.1).

In [101], Guhr et al. divide the history of random matrix theory into three epochs:

(i) The early period starting with Wigner's early work in the 1950's to mid 1960's, where basic classification results and a number of quantitative probes of spectra are established.

(ii) The middle period from mid 1960's to about 1983, where more data were gathered in various fields of physics and the range of applicability of random matrix theory was better established. Fundamental breakthroughs were not as common in this epoch and the analytical questions arising from the data coming from nuclear theory brought the need for new mathematical tools. One notable idea in this period, which we will discuss in more detail, is Balian's information theoretical approach to random matrix theory.

(iii) The period from 1983 to present, where random matrix theory enjoys a renewed interest, a flood of data and an increasing arsenal of analytical tools. More specifically random matrix models of quantum chromodynamics are a product of the last decade entirely.

The beginning of this last epoch is marked by two almost simultaneous events: the introduction of the supersymmetric method by Efetov [76] in 1983 and the Bohigas-Giannoni-Schmidt conjecture [36] mentioned in the introduction, stating the universality of random matrix spectra among chaotic quantum systems, in 1984.

Bohigas, Giannoni and Schmidt published a numerical study of the spectrum of the Schrodinger equation inside a 'box' of the shape of the reduced Sinai billiard. This quantum billiard problem is a generic example of a quantum mechanical system whose classical dynamics is chaotic. Bohigas et al. presented numerical evidence that the nearest neighbor spacing distribution of this system is identical to that of the gaussian orthogonal ensemble. Because of these numerical results they conjectured that local spectral characteristics of a quantum mechanical system whose classical dynamics is chaotic are given by a corresponding random matrix theory. (They specifically considered K-systems which are the most strongly mixing classical systems.)

The content of the Bohigas-Giannoni-Schmidt conjecture has some contrast with the early empirical agreement between random matrix theory and nuclear spectra. In the case of nuclear spectra the system under study is a many-body system whereas in the case of quantum billiards there are only 2 degrees of freedom. As seen from many other examples, including the hydrogen atom in a magnetic field [89], what really determines the validity of random matrix theory is the classical chaoticity of the problem, rather than the number of degrees of freedom of the system.

Although verified by various studies the Bohigas-Giannoni-Schmidt conjecture still remains unproven. For a further discussion of the status of this conjecture we refer the reader to [101]. Most of the attempts to link random matrix theory with quantum chaos rely on the semi-classical approximation. For a more detailed discussion of quantum chaos we refer the reader to [105].

Another important development at almost the same time was the introduction of the supersymmetry method in random matrix theory by Efetov in 1983 [76]. This has provided additional analytical tools to study theoretical problems associated with random matrix theory. We will further discuss this method in section 2.4 below.

Random matrix theory currently enjoys a renewed interest and it has been applied to many systems outside the scope of the nonrelativistic Schrodinger

equation as mentioned in the introduction. We will especially be interested in random matrix models relating to quantum chromodynamics as discussed in chapter 4.

2.2 Statistics of the eigenvalues

In this section we briefly outline some of the statistical measures used in probing the spectra of random matrix ensembles. Below we will start with a given probability density of the eigenvalues, $P_N(x_1, \ldots, x_N)$. For canonical ensembles it was given by (2.9),

$$P_{N,\text{canonical}}(x_1, \ldots, x_N) = \Delta^\beta(x) \exp\left(-N\beta \sum_{k=1}^{N} x_k^2\right). \tag{2.12}$$

Given a probability density, $P_N(x_1, \ldots, x_N)$, of the eigenvalues, x_l, of a matrix, H, perhaps the most important corresponding spectral measure is the *spectral density*, which can be obtained by successive integrations of the eigenvalues

$$\begin{aligned} \rho(x) &\equiv \langle \operatorname{Tr} \delta(x - H) \rangle \\ &= \left\langle \sum_k \delta(x - x_k) \right\rangle \\ &= N \int dx_2 \ldots dx_N P_N(x_1, \ldots, x_N), \end{aligned} \tag{2.13}$$

where the brackets denote an ensemble average.

More generally, one may study the higher correlation functions of the eigenvalues

$$R_k(x_1, \ldots, x_k) = \frac{N!}{(N-k)!} \int dx_{k+1} \ldots dx_N P_N(x_1, \ldots, x_N), \tag{2.14}$$

which is the probability density of finding one eigenvalue in the vicinity of each x_l, for $l = 1, \ldots, k$.

An alternative form of (2.14), which will be used in chapter 4 is obtained by realizing that a delta function may be represented as

$$\delta(x) = \frac{1}{\pi} \operatorname{Im} \frac{1}{x - i\varepsilon}. \tag{2.15}$$

Using this identity and the representation of $R_k(x_1, \ldots, x_k)$ in a similar form as in (2.13), we arrive at the following representation

$$R_k(x_1, \ldots, x_k) = \frac{1}{\pi^k} \int d[H] P_N(H) \prod_{l=1}^{k} \mathrm{Im}\mathrm{Tr} \frac{1}{x_l - i\varepsilon - H}. \tag{2.16}$$

As we will see below, this representation will be especially important in the supersymmetric formulation of random matrix theory. Sometimes it is more convenient to work with pseudo-correlators,

$$\hat{R}_k(x_1, \ldots, x_k) = \frac{1}{\pi^k} \int d[H] P_N(H) \prod_{l=1}^{k} \mathrm{Tr} \frac{1}{x_l - i\varepsilon - H}, \tag{2.17}$$

from which the correlation functions, R_k, can be constructed by taking the imaginary part of each trace.

In a number of situations one has to work with a portion of the spectrum in which the mean eigenvalue spacing is not constant. As many local spectral characteristics scale with the mean eigenvalue spacing one wishes to eliminate this dependence in those cases by a procedure called the *unfolding procedure* [215].

In order to do this one first obtains a smoothened spectral density, $\bar{\rho}$, from the original one, ρ. The unfolded spectrum is then given by

$$x_l^{\mathrm{unf}} = \int_0^{x_l} dx \bar{\rho}(x). \tag{2.18}$$

The statistical measures of the spectrum are defined in terms of the unfolded eigenvalues [149, 215, 101]. One such measure is the previously mentioned nearest neighbor spacing distribution.

Two other commonly used statistics are the number variance, Σ_2, and the Δ_3 statistic. They are defined in terms of the moments, M_k, of the number of levels in a fixed length of the spectrum while one moves this interval along the spectrum. Dividing the spectrum in to N intervals with an average number of n levels in each, the moments of the spectrum are defined by:

$$M_k(n) = \frac{1}{N} \sum_{l=1}^{N} n_l^k, \tag{2.19}$$

where n_l denotes the number of levels in the lth interval. Naturally $M_1(n)$ approaches n for a large sample.

The number variance is given by the second moment

$$\Sigma_2(n) = M_2(n) - n^2 \tag{2.20}$$

and the Δ_3 statistic is defined by an integral of the number variance

$$\Delta_3(L) = \frac{2}{L^4}\int_0^L dr\Sigma_2(r)(L^3 - 2L^2 + r^3). \tag{2.21}$$

2.3 Orthogonal polynomial method

For random matrix models whose partition functions are invariant under unitary transformations, a powerful method can be employed to compute the eigenvalue correlators and other local variables. This technique, known as the orthogonal polynomial method, expresses the Vandermonde determinant in (2.9) in terms of the orthogonal polynomials corresponding to the weight function of the ensemble. The partition function of eigenvalues in (2.14) can then be recursively integrated using orthogonality relations.

Some basic properties of and some useful identities for orthogonal polynomials are summarized in the appendix.

In this section we will only consider invariant unitary ensembles. Orthogonal and symplectic ensembles do not permit a direct application of the orthogonal polynomial for reasons that will become clear. However, they can be studied by so-called skew-orthogonal polynomials. This process, pioneered by Dyson [67] and Mahoux and Mehta [143], is much more involved and is discussed in more detail in chapter 4, where we introduce a method to relate results about unitary ensembles to corresponding orthogonal and symplectic ensembles.

We start by re-writing the equation (2.10) for the Vandermonde determinant in terms of the orthogonal polynomials, $p_l(x)$, corresponding to the weight function, $e^{-V(x)}$, of the random matrix ensemble given in (4.3). For the gaussian unitary ensemble $V(x) = x^2$ and one gets Hermite polynomials as discussed in the appendix. For chiral gaussian unitary ensemble it is customary to work with the squares of eigenvalues, $x_k = \lambda_k^2$, therefore x is in the interval $(0, \infty)$ and the weight function is given by $e^{-V(x)} = x^a e^{-x}$ and the corresponding orthogonal polynomials are Laguerre polynomials. The orthogonal polynomial method can be introduced for an arbitrary weight function even though the corresponding polynomials might be unfamiliar.

By adding columns of lower order polynomials, the Vandermonde determinant in (2.10) can be written as

$$\Delta(x) = \begin{vmatrix} p_0(x_1) & \cdots & p_0(x_N) \\ \vdots & & \vdots \\ p_{N-1}(x_1) & \cdots & p_{N-1}(x_N) \end{vmatrix}, \tag{2.22}$$

where we have assumed that the polynomials, $p_k(x)$, are chosen to be monic.

Introducing the (unnormalized) 'wave functions'

$$\varphi_k(x) = p_k(x)e^{-\frac{1}{2}V(x)} \tag{2.23}$$

one can write the probability density for the eigenvalues in (2.9) as

$$\begin{aligned} P_N(x_1,\ldots,x_N) &= \Delta^2(x)\exp\left(-NV(x)\right), \\ &= \begin{vmatrix} \varphi_0(x_1) & \cdots & \varphi_{N-1}(x_1) \\ \vdots & & \vdots \\ \varphi_0(x_N) & \cdots & \varphi_{N-1}(x_N) \end{vmatrix} \times \begin{vmatrix} \varphi_0(x_1) & \cdots & \varphi_0(x_N) \\ \vdots & & \vdots \\ \varphi_{N-1}(x_1) & \cdots & \varphi_{N-1}(x_N) \end{vmatrix}, \end{aligned} \tag{2.24}$$

which can then be expressed in terms of the *kernel* of the correlation functions

$$K_N(x,y) = \sum_{k=0}^{N-1} \frac{1}{h_k}\varphi_k(x)\varphi_k(y), \tag{2.25}$$

where the h_k are the normalization coefficients defined in the appendix. Substituting the definition of the kernel in (2.25), the probability density function for the eigenvalues can be written in the form of a determinant

$$P_N(x_1,\ldots,x_N) = \left(\prod_{k=0}^{N-1} h_k\right) \det\left[K_N(x_k,x_l)\right]_{k,l=1,\ldots,N}. \tag{2.26}$$

Using the orthogonality relations among $p_k(x)$ it is easy to prove the following identities

$$\int dx\; K_N(x,x) = N, \tag{2.27}$$

$$\int dz\; K_N(x,z)K_N(z,y) = K_N(x,y). \tag{2.28}$$

This identities can be used to recursively integrate the eigenvalues in (2.13) and (2.14). Finally we arrive at the following expressions for the spectral density and the eigenvalue correlation functions

$$\rho(x) = K_N(x,x), \tag{2.29}$$

$$R_k(x_1,\ldots,x_k) = \det\left[K_N(x_i,x_j)\right]_{i,j=1,\ldots,k}. \tag{2.30}$$

These relations can be simplified further by realizing that the right hand side of (2.25) is the left hand side of the Christoffel-Darboux identity given in

the appendix (up to an overall exponential factor). This allows one to express the kernel in terms of high order polynomials which can then be studied using asymptotic relations.

As discussed in the appendix, in the case of the chiral gaussian unitary ensemble the relevant orthogonal polynomials are the Laguerre polynomials, whose asymptotic forms are given by (A.17). This allows one to express the spectral properties of chiral gaussian unitary ensemble in the large N limit and in the microscopic region (discussed below) in terms of the Bessel kernel

$$K(x,y) = \frac{\sqrt{xy}}{x^2-y^2}\left(xJ_a(2y)J_{a+1}(2x) - yJ_a(2x)J_{a+1}(2y)\right) \ . \tag{2.31}$$

Remarkably analytical results of this type have been shown to agree with numerical studies from lattice QCD computations as we will discuss in more detail in chapter 4.

2.4 Supersymmetry method in random matrix theory

In this section we introduce another method used in random matrix theory studies. The supersymmetric formulation introduced below is used extensively in chapter 4 and unlike the orthogonal polynomial method it can be utilized in cases where the unitary invariance of the partition function is broken. Some of the necessary background for superanalysis is given in the appendix. Further information can be found in [32, 216, 95]. In this section we adopt the notation from [95].

Our aim will again be the computation of the higher correlation functions and we will start from the expression given in (2.16).

We will first obtain some simple identities for the trace of the inverse of an $N \times N$ matrix A:

$$\begin{aligned} \mathrm{Tr}\,\frac{1}{A} &= \frac{1}{2}\frac{\partial}{\partial j}\bigg|_{j=0} \frac{\det(A+j)}{\det(A-j)}, \\ &= \frac{1}{2}\frac{\partial}{\partial j}\bigg|_{j=0} \int d[\phi]d[\chi] \exp\left(i\phi^\dagger(A-j)\phi + i\chi^\dagger(A+j)\chi\right), \end{aligned} \tag{2.32}$$

where ϕ and χ are bosonic and fermionic vectors of length N, respectively and the integration measure is given by $d[\phi] = \prod d\phi_k^* d\phi_k$ and $d[\chi] = \prod d\chi_k^* d\chi_k$. The reader is referred to the appendix for our integration conventions for superintegrals.

Using (2.32) one can write (2.17) as a superintegral. More specifically, for the gaussian unitary ensemble where

$$P_{N,\mathrm{gue}}(H) = \frac{2^{N(N-1)/2}}{(2\pi\nu^2)^{N^2/2}} \exp\left(-\frac{1}{2\nu^2}\mathrm{Tr}\, H^2\right), \tag{2.33}$$

with variance, ν, one arrives at the identity

$$\hat{R}_k(x_1,\ldots,x_k) = \frac{1}{(2\pi)^k}\frac{\partial^k}{\prod_{i=1}^k \partial j_i} Z_k(j_1,\ldots,j_k)\Big|_{j_i=0}, \tag{2.34}$$

where the generating function, Z_k, is given by

$$Z_k(j_1,\ldots,j_k) = \int d[H]P_N(H) \int d[\Psi] \exp\left(2i\Psi^\dagger(\mathbf{x}+i\varepsilon\mathbf{H}+\mathbf{j})\Psi\right), \tag{2.35}$$

where Ψ is a $(kN+kN)$ supervector and we have introduced the following $(kN+kN)$ block-diagonal supermatrices

$$\begin{aligned} \mathbf{x} &= \mathrm{diag}\,(\,(x_1,\ldots,x_k)\otimes\mathbf{1_N}\,;\,(x_1,\ldots,x_k)\otimes\mathbf{1_N}\,), \\ \mathbf{H} &= \mathrm{diag}\,(\,\mathbf{1_k}\otimes H\,;\,\mathbf{1_k}\otimes H\,) \\ \mathbf{j} &= \mathrm{diag}\,(\,(j_1,\ldots,j_k)\otimes(-\mathbf{1_N})\,;\,(j_1,\ldots,j_k)\otimes\mathbf{1_N}\,). \end{aligned} \tag{2.36}$$

Here a semicolon separates bosonic blocks from fermionic ones.

The advantage of this formulation is that the H-integrals are now gaussian integrals which can be carried out analytically for arbitrary N. This introduces some fourth order terms in Ψ which can then be decoupled by introducing a $(k+k)$ supermatrix, σ. Eventually the remaining Ψ-integrals (which are now gaussian) can be done analytically and we are left with an expression for the generating function, Z_k, in (2.35), which involves only the newly introduced supermatrix, σ,

$$Z_k(j_1,\ldots,j_k) = 2^{k(k-1)} \int d[\sigma] e^{-\mathrm{str}\,\sigma^2} \mathrm{sdet}^{-N}\,(x-i\varepsilon-\sigma+j). \tag{2.37}$$

Here we have introduced the following $(k+k)$ supermatrices:

$$\begin{aligned} x &= \mathrm{diag}(\,x_1,\ldots,x_k\,;\,x_1,\ldots,x_k\,), \\ j &= \mathrm{diag}(\,-j_1,\ldots,-j_k\,;\,j_1,\ldots,j_k\,). \end{aligned} \tag{2.38}$$

The dimension, N, of the original matrices have now completely disappeared from the integrations except for being a parameter. This procedure, which resembles a Fourier transform, is known as the Hubbard-Stratonovitch

transformation. Despite small changes it can be adopted even in a case where the unitary invariance of the ensemble is broken. One notable restriction is the use of gaussian weight functions. For a discussion in the case of non-gaussian weight functions we refer the reader to [106].

The expressions (2.34) and (2.37) can be treated either in the saddle-point approximation or exactly for any N, which for the gaussian unitary ensemble leads to the well known correlator kernel in terms of Hermite polynomials. The details of this procedure are involved. They can be found in [95] and are also illustrated in chapter 4 for the case of the chiral gaussian unitary ensemble, where one instead recovers the Laguerre kernel.

2.5 Balian's information theoretical approach to random matrix theory

In 1968 Balian [18] has shown that the canonical random matrix ensembles of Wigner and Dyson can be realized as 'minimal information models' defined in an appropriate sense within the context of Shannon's information theory [179, 138]. (A little earlier, Bronk [49] also referred to canonical ensembles as 'most random ensembles' in an information theoretical sense.) This perspective, outlined below, is not of major practical computational value in random matrix theory. However it is important for putting universality studies in the proper context.

The amount of information contained in a probability density, $\{p_k\}$, for discrete events, $k = 1, \ldots, N$, is given by

$$I[p] \equiv \sum_k p_k \ln p_k. \tag{2.39}$$

This equation resembles the definition of entropy in statistical mechanics except for a minus sign. Furthermore, it is easy to see that the minimum of $I[p]$ is obtained for $p_k = \frac{1}{N}$, for all k, which is naturally recognized as the case where least is known about which outcome will happen. The maximum information is attained when $p_k = \delta_{kn}$, for some n, which is the case where we know the outcome with absolute certainty. Hence, the formal definition for information given in (2.39) also makes intuitive sense.

A natural generalization of (2.39) to continuous probability distributions is given by

$$I[p] \equiv \int dx p(x) \ln p(x), \tag{2.40}$$

which can naturally be used with probability distributions over spaces of matrices, $P(H)$, as discussed in previous sections.

As Balian noticed, the probability distribution in (2.8) for any of the three canonical ensembles can be obtained by minimizing the information content of $P(H)$ under the constraint of a fixed expectation value for $\langle \mathrm{Tr}\, H^2 \rangle$. (An implicit constraint is the Dyson index, β, of the ensemble.)

In general, for an ensemble satisfying constraints of the type

$$\langle f_k(H) \rangle = \int d[H] P(H) f_k(H) = c_i, \tag{2.41}$$

the constrained minimization of the information content gives the probability density

$$P(H) = \alpha \exp\left(\sum_k \lambda_k f_k(H) \right), \tag{2.42}$$

where λ_k are Lagrange multipliers chosen to satisfy (2.41). Hence the random matrix ensemble can be 'dialed' to satisfy any constraints of type (2.41). Such an example is an ensemble having a fixed level density [149]:

$$\langle \mathrm{Tr}\, \delta(H - x) \rangle = \rho_0(x). \tag{2.43}$$

In this case

$$\begin{aligned} P(H) &= \alpha \exp\left(\int dx \lambda(x) \mathrm{Tr}\, \delta(H - x) \right), \\ &= \alpha \exp\left(\mathrm{Tr}\, \lambda(H) \right), \end{aligned} \tag{2.44}$$

where $\lambda(x)$ are determined in such a way as to satisfy (2.43).

Following these examples a picture emerges, where random matrix ensembles are viewed as models with the least amount of detail (or with the greatest simplicity) after a given number of constraints are satisfied. This allows one to study the consequences of individual constraints without interference from other artifacts which may belong to some specific model satisfying the constraint, but not the other.

It shall also be noted that not all constraints carry important spectral information. As an example, we shall see in chapter 4 that the local spectral properties are highly independent of the exact form of $P(H)$.

2.6 Universality in random matrix theories and an analogy to central limit theorem

Central to all random matrix studies lies the fact that they reproduce certain features of real physical spectra. This is in spite of the fact that random matrix models are extremely simple ('maximally simple' in the sense discussed in the previous section.) constructions. The reason behind this is universality.

As outlined by the Bohigas-Giannoni-Schmidt conjecture not all details of a (quantum) system play a role in the determination of (local) spectral properties. Hence many systems sharing similar symmetry properties enjoy the same local spectral statistics (up to an unfolding process).

This is highly reminiscent of the central limit theorem which establishes the ubiquity of the gaussian distribution. Although no such formal analogy can be established there are some parallels. Most importantly, the central limit theorem provides an example where the final distribution of certain quantities are independent of the initial selection of some probability distributions. More precisely, the central limit theorem states that the distribution of the sum of random numbers drawn from a fixed probability distribution approaches a gaussian under a variety of conditions: i.e. the gaussian is invariant under deformations of the initial distribution.

Although they share many similarities in their perspective, (as we will see in chapter 4) universality theorems in random matrix theory are generally narrower in scope, where each generalization might require a special study. Therefore we think it useful to outline a proof of the central limit theorem to provide an example.

To illustrate the point we follow a proof given by Feinberg and Zee [81] using a renormalization group idea. For details we refer the reader to the original reference. For the use of renormalization group techniques in random matrix theory studies see [48, 81, 118, 117].

Consider N random numbers, $\{x_1, \ldots, x_N\}$ drawn independently from the same distribution $Q(x)$, which is normalized such that

$$\langle x \rangle = 0 \quad \left\langle x^2 \right\rangle = \frac{\sigma^2}{N}. \tag{2.45}$$

This normalization guarantees that for large N the sum, $\sum x_k$, has a finite variance independent of N.

We wish to study the distribution of the sum of these numbers, namely

$$P_N(s;\sigma) = \left\langle \delta(s - \sum_{k=1}^{N} x_k) \right\rangle_N . \tag{2.46}$$

We will proceed by comparing $P_{N+1}(s;\sigma)$ with $P_N(s;\sigma)$ and thereby obtaining a differential equation for it.

$$\begin{aligned} P_{N+1}(s;\sigma) &= \left\langle \delta(s - \sum_{k=1}^{N} x_k - x_{N+1}) \right\rangle_{N+1} . \\ &= \left\langle \delta(s - \sum_{k=1}^{N} x_k) \right\rangle_{N+1} \\ &\quad + \frac{1}{2}\frac{\sigma}{N+1}\frac{\partial^2}{\partial s^2} \left\langle \delta(s - \sum_{k=1}^{N} x_k) \right\rangle_{N+1} + \dots . \end{aligned} \tag{2.47}$$

Higher order terms are suppressed in the large N limit. Note that the variance changes with the increment in N according to

$$\sigma' = \left(\frac{N}{N+1}\right)^{\frac{1}{2}} \sigma. \tag{2.48}$$

Combining the last two equations we arrive at

$$N\frac{\partial P_N}{\partial N} = \frac{1}{2}\sigma^2 \frac{\partial^2 P_N}{\partial s^2} - \frac{1}{2}\sigma \frac{\partial P_N}{\partial \sigma}. \tag{2.49}$$

For $P_N(s,\sigma)$ to have a large N limit the left hand side of (2.49) should vanish which leaves us with

$$\left(\sigma \frac{\partial^2}{\partial s^2} - \frac{\partial}{\partial \sigma}\right) P_N(s,\sigma) = 0. \tag{2.50}$$

We need to relate σ and s to proceed further which is done by a scaling argument

$$P(s,\sigma) = \frac{1}{\sigma} P(\frac{s}{\sigma}, 1), \tag{2.51}$$

which leads to the following differential equation for $P(s,\sigma)$:

$$\left(\sigma^2 \frac{\partial^2}{\partial s^2} + s\frac{\partial}{\partial s} + 1\right) P(s,\sigma) = 0. \tag{2.52}$$

The normalized solution to (2.52) is the gaussian

$$P(s,\sigma) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{s^2}{2\sigma^2}\right), \tag{2.53}$$

which is the statement of the central limit theorem.

Chapter 3

The Dirac operator in quantum chromodynamics

Quantum chromodynamics (QCD) is believed to be a viable microscopic theory describing strong interactions. The Dirac operator carries important information about quantum chromodynamics, and the random matrix models introduced in the next chapter are modeled after it. In this chapter we will introduce some properties of the Dirac operator of quantum chromodynamics with an emphasis on the concepts used in chiral random matrix theory. Since quantum chromodynamics is too large in scope for a comprehensive review; we refer the reader to [52] for a more complete discussion.

The outline of this chapter is as follows. In the next section we will introduce quantum chromodynamics and some of its essential properties. In section 3.2 we will discuss the chiral symmetry of the Dirac operator. In section 3.3 we will derive the Banks-Casher relation, which provides the link between the eigenvalues of the Dirac operator and the chiral condensate. Finally in section 3.4 we will introduce the Leutwyler-Smilga sum rules, which are identities satisfied by the Dirac eigenvalues reproduced in chiral random matrix theories. Throughout this chapter we will work with Euclidean space-time.

3.1 Introduction to quantum chromodynamics

It was realized in the early sixties that the observed hadron spectrum obeys an approximate symmetry given by the group $SU(3)$. This was an extension of the already known isospin symmetry of SU(2). In 1961 Gell-Mann [91] and Ne'eman [157] proposed the *eightfold way* theory of strong interactions named after the octet representation of the $SU(3)$ group which was identified

in the hadron spectrum. One notable feature of this theory is that fundamental (triplet) representation of $SU(3)$ could not be identified with any observed particles. Hence it was postulated that all hadrons consist of fermions called *quarks*, which realize the fundamental representation of $SU(3)$. The three quark *flavors* are named *up*, *down* and *strange*. However the simple quark model has paradoxes. First of all, in order to explain the observed hadron spectrum quarks have to be fractionally charged. Therefore, due to charge conservation, the lowest quark state must be absolutely stable. Such a particle has never been observed directly. Furthermore the quark decomposition of certain hadrons seem to violate the spin-statistics theorem without the assumption of a further quantum number. It was therefore postulated that quarks posses a hidden degree of freedom, called *color*, which corresponds to another $SU(3)$ group.

It should be emphasized that the flavor $SU(3)$ group of the eightfold way has nothing to do with the color $SU(3)$. They correspond to totally different quantum numbers. First of all, the color $SU(3)$ is assumed to be exact while the flavor symmetry is approximate. Another difference is that the flavor symmetry is global while the color symmetry is local as we will discuss below. Furthermore, more quarks have since been found to increase the number of quark flavors to six: down, up , strange, charmed, bottom and top. The latter three are too heavy to play an important role in typical hadronic scales. A postulate of this theory is that only color singlet states are observed in nature. Hence free quarks are never observed. This is known as *color confinement*.

Quantum chromodynamics is a gauge theory [228]. It is obtained by making the global color $SU(3)$ symmetry local. Hence it has the general structure of a non-abelian Yang-Mills theory given by a local lagrangian

$$\mathcal{L}_{QCD} = \frac{1}{4}F_A^2 + \sum_{f=1}^{N_f} \bar{\psi}_f(\not{D} + m_f)\psi_f. \tag{3.1}$$

Here N_f is the number of quark flavors and the covariant derivative operator is defined by

$$D_\mu = \partial_\mu + i\lambda^a A_\mu^a, \tag{3.2}$$

where λ^a are the generators of the Lie algebra of $SU(3)$ and A_μ is the gauge potential. F_A is the field strength given in terms of the gauge potential, $F_A = DA$. In component form we have

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + f_{bc}^a A_\mu^b A_\nu^c, \tag{3.3}$$

where f^a_{bc} are the structure constants of $SU(3)$.

The *Dirac operator*, $\not{D}$, acts on the spinor (fermion) fields, ψ, and is given in terms of the covariant derivative operator

$$\not{D} = \gamma_\mu \left(\partial^\mu + i\lambda^a A^{a\mu}\right). \tag{3.4}$$

It is easily seen that Dirac operator is anti-hermitian. The Euclidean γ-matrices introduced in (3.4) satisfy the following anti-commutation relation

$$\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}. \tag{3.5}$$

In principle, it should be possible to derive all properties of hadrons from the lagrangian in (3.1) (except for the electroweak contributions and for gravity which is irrelevant at the energy scales in question). However this has proved to be an immensely difficult task due to the nonlinear nature of the theory. Therefore it is sometimes more beneficial to work with an effective theory of quantum chromodynamics. For example, in the low energy limit of QCD and for light quarks this would be the Weinberg lagrangian [215]:

$$\mathcal{L}_{\text{eff}} = \frac{F_\pi^2}{4}\text{Tr}\,\left(\partial_\mu U \partial^\mu U\right) - \frac{\Sigma}{2}\text{Tr}\,\left(\mathcal{M}U + \mathcal{M}U^\dagger\right). \tag{3.6}$$

Here $U(x)$ are $SU(N_f)$ valued fields, F_π is the pion decay constant and Σ is the chiral condensate to be introduced below. In (3.6) $U(x)$ is given in terms of the pion fields, $\Pi_a(x)$,

$$U(x) = \exp\left(\frac{i\sqrt{2}}{F_\pi}\Pi_a(x)\lambda^a\right). \tag{3.7}$$

Expanding (3.6) in terms of the pion fields one obtains in the lowest order (and for equal quark masses)

$$\mathcal{L}_{\text{eff}} = \frac{1}{2}\partial_\mu \Pi^a \partial^\mu \Pi^a + \frac{\Sigma m}{F_\pi^2}\Pi^a \Pi^a. \tag{3.8}$$

Reading the pion mass from this effective lagrangian we arrive at the Gell-Mann - Oakes - Renner relation [92] between the pion and quark masses

$$m_\pi^2 = \frac{2\Sigma m}{F_\pi^2}. \tag{3.9}$$

A possible addition to the QCD lagrangian in (3.1) is a topological term:

$$\mathcal{L}_{QCD} = \frac{1}{4}F_A^2 + \sum_{f=1}^{N_f}\bar{\psi}_f(\not{D} + m_f)\psi_f + \frac{i\theta}{32\pi^2}F\tilde{F}. \tag{3.10}$$

Here $\tilde{F}$ denotes the Hodge dual of F given by

$$F^{\mu\nu} = \frac{1}{2}\epsilon_{\rho\sigma\mu\nu}F^{\rho\sigma}, \tag{3.11}$$

where $\epsilon_{\rho\sigma\mu\nu}$ is the totally antisymmetric tensor of rank 4. A notable feature of this additional term is that it is a total derivative. Hence it does not effect the classical equations of motion. The integral of the factor multiplying θ can be identified as the winding number, ν, of the underlying gauge field. It is a topological invariant. The θ-term in (3.10) violates P and CP invariance [52]; however the actual value of θ is not known.

The Euclidean partition function of quantum chromodynamics is given by a path integral of the QCD action defined in (3.1)

$$Z_{QCD} = \int \mathcal{D}A \int \mathcal{D}\psi \exp\left[-\int d^4x \left(\sum_f \bar{\psi}_f(\not{D} + m_f)\psi_f + \frac{1}{4}F_A^2\right)\right]. \tag{3.12}$$

The fermion fields can be integrated out analytically. This results in the following average over gauge field configurations:

$$Z_{QCD} = \int \mathcal{D}A e^{-\int d^4x \frac{1}{4}F_A^2} \prod_f \det(\not{D} + m_f). \tag{3.13}$$

3.2 Chiral symmetry of the Dirac operator

The Dirac operator defined in (3.4) carries information between the gauge fields and the fermion fields. Therefore its study is essential for a proper understanding of quantum chromodynamics. In the following sections and in chapter 4 we will be especially interested in the spectrum of the Dirac operator. Since $\not{D}$ is an anti-hermitian operator, its eigenvalues are purely imaginary:

$$\not{D}\phi_k = i\lambda_k\phi_k. \tag{3.14}$$

The axial symmetry of the Dirac operator is expressed by

$$\{\gamma_5, \not{D}\} = 0, \tag{3.15}$$

where $\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4$. As we will see shortly, this relation has important consequences on the Dirac spectrum. Most notably it can be easily seen that all nonzero eigenvalues come in pairs $\pm\lambda_k$. If ϕ_k is an eigenvector as defined by (3.16), then $\gamma_5\phi_k$ is also an eigenvector with an opposite eigenvalue:

$$\not{D}(\gamma_5\phi_k) = -\gamma_5\not{D}\phi_k = -i\lambda_k\gamma_5\phi_k. \tag{3.16}$$

If $\lambda_k = 0$ then ϕ_k and $\gamma_5\phi_k$ need not be linearly independent.

The chiral structure of the Dirac operator is more easily seen in the chiral representation of γ-matrices: (*Chirality* is a term of Greek origin representing *handedness*)

$$\gamma_k = \begin{pmatrix} 0 & i\sigma_k \\ -i\sigma_k & 0 \end{pmatrix}, \quad \gamma_4 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{3.17}$$

In this basis we have

$$\gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3.18}$$

The block structure above can be identified with fermion modes of opposite handedness. More precisely, we will introduce the left-handed and right-handed modes as eigenvectors of the γ_5 operator:

$$\begin{aligned} \gamma_5\phi_R &= +\phi_R, \\ \gamma_5\phi_L &= -\phi_L. \end{aligned} \tag{3.19}$$

The QCD partition function, introduced in (3.12) is invariant under chiral rotations for vanishing quark masses. Therefore one expects that the hadron spectrum shows this symmetry approximately. (Real quark masses are nonzero but small compared to the typical hadronic scale.) However this is not the case: we do not observe opposite parity partners of the hadrons that are known to exist. Thus the chiral symmetry is spontaneously broken.

It is expected phenomenologically that the chiral symmetry is restored at sufficiently high temperature or baryon density [107]. The order parameter of the chiral phase transition is the *chiral condensate*:

$$\Sigma \equiv \lim_{m\to 0} \lim_{V_4\to\infty} \frac{1}{N_f V_4} \frac{\partial}{\partial m} \ln Z_{QCD}(m). \tag{3.20}$$

Here V_4 is the 4-volume of space-time. It is easily seen that it can be written in the form of an amplitude for the creation of left-handed modes and the destruction of right-handed ones and vice versa:

$$\Sigma = \left|\left\langle \bar{\Psi}\Psi \right\rangle\right| = \left\langle \bar{\Psi}_L\Psi_R + \bar{\Psi}_R\Psi_L \right\rangle. \tag{3.21}$$

The value of Σ in vacuum is expected from phenomenological arguments to be around $(240\text{MeV})^3$.

3.3 The Banks-Casher relation

In this section we will derive a relation between the chiral condensate and the spectrum of the Dirac operator, known as the Banks-Casher relation [22].

We start by rewriting the QCD partition in terms of Dirac eigenvalues. The fermion determinant in (3.13) can be expressed (for vanishing topological charge) as

$$\det(\not{D} + m) = \prod_k{}' (\lambda_k^2 + m^2), \tag{3.22}$$

where the product is over positive eigenvalues only. (We have used the fact that nonzero eigenvalues come in opposite pairs.) Combining this with (3.20) results in the following expression for the chiral condensate

$$\Sigma = \lim_{m\to 0} \lim_{V_4\to\infty} \frac{1}{V_4} \left\langle \sum_k{}' \frac{2m}{\lambda_k^2 + m^2} \right\rangle . \tag{3.23}$$

The sum is over the nonzero eigenvalues and the brackets denote an average over all gauge field configurations.

The order of limits in (3.23) is important. It is essential that the thermodynamic limit is taken before the chiral limit. In the thermodynamic limit the Dirac eigenvalues fill the region near zero virtuality. (Virtuality is a term referring to Dirac eigenvalues.) Hence a continuous spectral density can be defined in the continuum limit. However if the order of the limits were to be interchanged (3.23) would become identically zero.

In order to proceed we observe that the argument of the sum in (3.23) is the representation of a delta function in the limit $m \to 0$. Hence we can rewrite (3.23) as

$$\Sigma = \frac{\pi}{V_4} \rho(0), \tag{3.24}$$

where the spectral density, $\rho(\lambda)$, is defined by

$$\rho(\lambda) = \langle \delta(\lambda - \lambda_k) \rangle . \tag{3.25}$$

The brackets once again denote the gauge field average.

The relation (3.24) is known as the Banks-Casher formula. It states that for small quark masses the order parameter of the chiral phase transition is given by the microscopic part of the spectral density of the Dirac operator. This is of essential importance to chiral random matrix theory because the microscopic properties of the Dirac eigenvalues are universal as we shall investigate below.

3.4 Leutwyler-Smilga sum rules

Leutwyler-Smilga sum rules are closed expressions for $V^{-2n}\sum_k' \lambda_k^{-2n}$ for integer values n whenever the sum converges. (Similar sum rules involving multiple eigenvalues are also possible but not discussed here.) One remarkable property of this infinite set of relations among Dirac eigenvalues is the fact that they can be derived from chiral random matrix theories [180].

Leutwyler and Smilga [141] have studied the the chiral lagrangian in (3.6) in the limit when the pion Compton wavelength is larger than the typical dimensions of the box:

$$m_\pi << \frac{1}{L} \tag{3.26}$$

In this limit the effective partition function for QCD with nonzero quark masses and an arbitrary θ-angle is given by

$$Z_{\text{eff}}(\mathcal{M},\theta) = \int d[U] \exp\left(V_4 \Sigma \text{ReTr}\, \mathcal{M} U e^{i\theta N_f}\right). \tag{3.27}$$

Here the integral is over $SU(N_f)$ with respect to its Haar measure.

The effective partition function for a given topological sector is obtained by a Fourier transform

$$\begin{aligned} Z_{\text{eff},\nu}(\mathcal{M}) &= \frac{1}{2\pi}\int_0^{2\pi} e^{-i\nu\theta} Z_{\text{eff}}(\mathcal{M},\theta) \\ &= \int_{U\epsilon U(N_f)} d[U] \text{det}^\nu(U) \exp\left(V_4 \Sigma \text{ReTr}\, \mathcal{M} U\right). \end{aligned} \tag{3.28}$$

Before proceeding further we would like to point out that the last integral can be recognized as a 'chiral' version of an Itzykson-Zuber-Harish-Chandra integral. It has been part of this thesis work [124] to develop the techniques for the analytical computation of these integrals. We will discuss this in detail in chapter 4.

The simplest Leutwyler-Smilga sum rule is obtained by expanding Z_ν to leading order in m^2 [215]:

$$Z_{\text{eff},\nu=0}(m) = 1 + m^2 \frac{\Sigma^2 V^2}{4}. \tag{3.29}$$

The expansion of (3.13) gives another such relation

$$\frac{Z_{\text{QCD},\nu=0}(m)}{Z_{\text{QCD},\nu=0}(0)} = 1 + m^2 N_f \left\langle \sum{}' \frac{1}{\lambda_k^2} \right\rangle. \tag{3.30}$$

Comparing the coefficients of m^2 we arrive at

$$\frac{1}{V^2}\left\langle {\sum}' \frac{1}{\lambda_k^2} \right\rangle = \frac{\Sigma^2}{4N_f}. \tag{3.31}$$

This is the first Leutwyler-Smilga sum rule. Others are obtained by expanding to higher orders in m^2 (or for different quark masses). Remarkably this set of relations are also satisfied by the eigenvalues of the random matrix Dirac operator as mentioned above. It shall be noted that they do not determine the Dirac spectrum uniquely.

Chapter 4

Universality in random matrix theories of QCD

This chapter will consist mostly of the work we have published during the course of this thesis study. We will first begin with an introduction to chiral random matrix ensembles. Then we will list some of the important characteristics of these ensembles. In section 4.3 we will present an early study of the universality of the microscopic part of the spectral density. In section 4.4 we will introduce a class of group integrals, called Harish-Chandra-Itzykson-Zuber integrals. They will be important in the next two sections where we first develop chiral generalizations of these integrals and then apply them to the computation of higher correlation functions. In section 4.7 we will review a universality proof for invariant unitary ensembles, which we will refer to in the last section. In section 4.8 we will develop a framework to relate spectral information about unitary ensembles to corresponding orthogonal and symplectic ensembles, which are harder to treat analytically.

4.1 Introducing chiral random matrix theories

We have seen in the previous chapter that the Euclidean partition function for QCD is given by

$$Z_{QCD} = \int \mathcal{D}A e^{-\int d^4x \frac{1}{4}F_A^2} \prod_f \det(\not{D} + m_f), \tag{4.1}$$

where $\not{D}$ is the Dirac operator.

The chiral structure of the Dirac operator, discussed in section 3.2, can be made more transparent in a chiral basis. In such a basis $\not{D}$ has the following

block structure:

$$\not{D} \equiv \not{\partial} + i\not{A} = i \begin{pmatrix} 0 & \mathbf{C}^\dagger \\ \mathbf{C} & 0 \end{pmatrix} \tag{4.2}$$

where $\mathbf{C}$ is (for $SU(3)$ gauge fields) a complex matrix whose exact form depends on the gauge field configuration in the background.

Ordinarily the gauge field averages in (4.1) cannot be carried out analytically. This has led to the study of effective field theories of QCD as discussed above. A random matrix model of QCD may be viewed as one such model where the precise gauge field average in (4.1) is replaced by a random non-local interaction

$$Z_{RMT} = \int \mathcal{D}[\mathbf{C}] e^{-\mathrm{tr}\, V(\mathbf{C}^\dagger \mathbf{C})} \prod_f \det(\not{D} + m_f) \tag{4.3}$$

where the Dirac operator, $\not{D}$, is still given by (4.2) and the measure $\mathcal{D}[\mathbf{C}] = \prod d\mathbf{C}_{ij} d\mathbf{C}^*_{ij}$ is over all independent degrees of freedom of $\mathbf{C}$. However in this context the matrix entries of $\mathbf{C}$ are understood to be random numbers weighed with respect to $\exp(-\mathrm{tr}\, V(\mathbf{C}^\dagger \mathbf{C}))$ and their connection to an external gauge field is given up entirely.

Because of the block structure of $\not{D}$ in (4.2) these ensembles are known as the chiral ensembles. We have seen in chapter 3 that this block structure implies that all nonzero Dirac eigenvalues come in pairs, $\pm\lambda_k$. We will see later that this additional constraint (which is not present, for example, in standard Dyson ensembles) dictates some profound properties for the spectrum of the Dirac operator, including the universality of the distribution of the microscopic Dirac eigenvalues.

The case $V(x) = x$ in (4.3) corresponds to the chiral gaussian ensembles which are easier to handle computationally. However as we will see below many interesting results valid for gaussian ensembles carry over to non-gaussian or even non-(unitary-)invariant ensembles. This kind of 'universality' parallels the central limit theorem in its content as discussed in chapter 2; however it has no direct bearing on the validity of random matrix theory results for *any* ensemble to the actual spectrum of QCD. Such a connection as well as the domain of validity of random matrix theory in QCD has been studied recently [165], where a Thouless energy was identified below which random matrix theory results agree with quantum chromodynamics.

Following the definitions for original Dyson ensembles, chiral ensembles have been named according to the degrees of freedom per matrix element, β. Above example, namely QCD with three colors and fundamental fermions,

corresponds to $\beta = 2$ and is known as the *chiral gaussian unitary ensemble*. QCD with two colors and fundamental fermions admits an additional anti-unitary symmetry which lets one to express the entries of the Dirac operator by real numbers (up to an overall factor i), and this case of $\beta = 1$ is known as the *chiral gaussian orthogonal ensemble*. Finally, for adjoint fermions or for two colors and staggered fermions the QCD Dirac operator can be described by a quaternion real matrix (again up to a factor i), which has the Dyson index $\beta = 4$, corresponding to the *chiral gaussian symplectic ensemble*.

The description of the chiral ensemble in (4.3) with random non-local interactions suggest it to be similar to mean field theory. Chiral random matrix theory does not carry any local information about QCD. Even the dimensionality of space-time or Lorenz invariance only enters through the existence and symmetries of γ-matrices, which dictate the subsequent symmetry structure of the Dirac operator, $\not{D}$. It turns out, however, that the additional information in chiral random matrix theory (the corresponding chiral and anti-unitary symmetries) has implications that go far beyond mean field theory.

More specifically we will see below that chiral random matrix theory spectrum in the microscopic limit coincides with the corresponding spectrum in QCD. The Banks-Casher relation, which relates these spectral data to the order parameter of the chiral phase transition , establishes a connection between chiral random matrix theory and (at least potentially) observable properties of QCD. The reader is cautioned however that, as we will see below, the exact form of the phase diagram of QCD cannot be obtained via random matrix theory studies. Such studies at most grant a qualitative picture of the phase diagram. It may be argued that the real power of chiral random matrix theory comes from the *exact* predictions about the microscopic Dirac spectrum which can be directly computed on lattice QCD simulations. In that context random matrix theory studies are complimentary to numerical studies.

A possible explanation for the aforementioned exact agreement for microscopic spectrum comes from the Bohigas-Giannoni-Schmidt conjecture discussed in chapter 2. It is known that strongly interacting gauge fields show classically chaotic behaviour.

One can consider many variations on the random matrix model defined in (4.3). We have already mentioned the possibility of varying the potential $V(x)$, which generally does not change the universality class of the spectrum (see section 4.7) and changing the Dyson index, β, which significantly effects the spectral properties. However, more profound variations are possible.

One such variation is to include information about the 'topological sector' of the Dirac operator (or rather its defining gauge field) in the corresponding

random matrix model. The Atiyah-Singer index theorem states that in the background of a gauge field with topological charge, ν, The Dirac operator has an excess of left handed zero modes over right handed ones (or vice versa) by an amount ν. This can be transferred to the random matrix model by choosing the random matrix

$$\not{D} = i \begin{pmatrix} 0 & \mathbf{C}^\dagger \\ \mathbf{C} & 0 \end{pmatrix} \tag{4.4}$$

to have rectangular blocks, $\mathbf{C}$, with dimension $N \times (N + \nu)$. It is easily seen that such a matrix has ν 'kinematical' zeroes.

A more interesting variation to (4.3) is to include a schematic temperature dependence in the partition function. This is done by using a lattice regularization of (4.1) with anti-periodic boundary conditions in the time direction to expand the fermion fields as a sum over Matsubara frequencies. Using this basis the Dirac operator can be written in such a way to include the temperature dependence. This is explained in detail in section 4.3. The model including only the lowest Matsubara frequency πT is given by

$$\not{D} = i \begin{pmatrix} 0 & \mathbf{C}^\dagger + \pi T \\ \mathbf{C} + \pi T & 0 \end{pmatrix} \tag{4.5}$$

where the partition function still retains the general form (4.3). An interesting feature of this model is that the weight function is no longer invariant under unitary transformations,

$$\begin{aligned} &\mathbf{C} \to U^\dagger \mathbf{C} V, \\ &U, V \epsilon U(N). \end{aligned} \tag{4.6}$$

This results in various computational difficulties: the group integrals in the partition function no longer factor out and the orthogonal polynomial method no longer works. Nevertheless this model will be examined in detail below as it provides a schematic tool to investigate the phase diagram of the chiral phase transition in QCD. The most important result about this model is that its local spectral distribution functions have the same functional form as the standard gaussian model except for a trivial temperature dependent scaling of the eigenvalues. This statement is valid under a certain critical temperature above which the microscopic part of the spectral density vanishes identically. Sections 4.3 and 4.6 will include detailed investigations of these properties.

In some of the studies below certain simplifications are made to (4.3), like taking the quark mass terms, m_f, to be identical or zero. Another simplification is to entirely ignore the fermion determinant $\prod \det(\not{D}+m)$ in the partition

function. This is called the 'quenched' approximation and a similar step is taken in lattice QCD studies as the computational cost increases dramatically when this determinant is taken into account in Monte-Carlo computations.

Now it is time for a further description of some of the properties of the spectra of chiral ensembles.

4.2 Some general properties of the spectra of chiral ensembles

We have already mentioned the most important property of the Dirac spectrum, namely that the spectrum is symmetric about the zero of the eigenvalue axis:

$$\not{D}\phi_k = \pm i\lambda_k \phi_k. \tag{4.7}$$

Hence one can give a complete description by working only with positive eigenvalues.

The spectrum of any of the chiral ensembles contains three regions with distinct behaviours: 'the hard edge', 'the bulk' and 'the soft edge'. The hard edge is the region containing microscopic eigenvalues or, using a more local term, the region near *zero virtuality*. Quantitatively, the hard edge covers the region $0 < \lambda < 1/\sqrt{N}$, where N is the dimension of the blocks of the Dirac operator and the eigenvalues are scaled in such a way that the largest is about one. The soft edge corresponds to the largest eigenvalues and the upper end of the spectrum. The bulk, on the other hand, covers everything that is sufficiently far away from both ends.

The overall shape off the spectrum for all three Gaussian ensembles (namely orthogonal, unitary and symplectic) is a semi-circle, the same as the original Dyson ensembles. This is an artifact of the Gaussian ensembles and the overall shape of the spectrum is not universal. However local spectral properties, such as the nearest neighbor spacing distribution, spectral correlation functions or (for the hard edge) the microscopic spectral density appear to be *universal* under very general conditions as long as one stays in the same spectral region and the same Dyson index, β.

In other words, for a given value of β and for a given eigenvalue domain (e.g. the hard edge) the functional form of the unfolded local spectral functions obtained for the Gaussian ensembles remains the same even if one were to replace the quadratic interaction $V(x) = x$ in (4.3) with an arbitrary polynomial. Even replacing the unitary invariant $V(C^\dagger C)$ with certain non-invariant

interactions may leave the aforementioned functional forms unchanged. The exact statements regarding these *universal* properties and their domain of validity is the topic of following sections as well as an integral part of this thesis work. Below we will summarize the results for Gaussian ensembles.

The bulk and the soft edge of chiral ensembles are precisely the same as the original Dyson ensembles introduced in chapter 2. The only domain indigenous to chiral ensembles is the hard edge. Fortunately, this is the most relevant domain for physical applications, thanks to the Banks-Casher relation.

Many of the local properties of the spectrum of (4.3) in the hard edge can be expressed in terms of the Bessel Kernel, which for vanishing quark mass terms can be written as

$$\begin{aligned} K_S(x,y) &= \lim_{N\to\infty} \frac{1}{2N} K_N(\frac{x}{N}, \frac{y}{N}) \\ &= \frac{\sqrt{xy}}{x^2-y^2}\left(xJ_{N_f}(2y)J_{N_f+1}(2x) - yJ_{N_f}(2x)J_{N_f+1}(2y)\right) \quad . \end{aligned} \tag{4.8}$$

Here K_N is the kernel of the correlation functions as introduced in chapter 3, where N denotes the size of the blocks of the Dirac operator. K_S denotes the microscopic limit of this kernel as defined by the limiting procedure and J_k are the Bessel functions.

In terms of this kernel the k-point correlation function of the eigenvalues, introduced in chapter 2, can be expressed as

$$R_k(x_1, \ldots, x_k) = \det[\Sigma K_S(x_p\Sigma, x_q\Sigma)] \tag{4.9}$$

where Σ denotes the chiral condensate and the microscopic spectral density can be expressed as

$$\begin{aligned} \rho_s(x) &= \lim_{y\to x} K_S(x,y) \\ &= x\left(J^2_{N_f}(2x) - J_{N_f+1}(2x)J_{N_f-1}(2x)\right). \end{aligned} \tag{4.10}$$

In the presence of a nonzero topological charge, ν, these equations still remain valid if one replaces N_f by $N_f + |\nu|$.

4.3 Universality of the spectral density near zero virtuality

This section contains a study of the microscopic part of the spectral density for a non-invariant deformation of the chiral gaussian unitary ensemble

as given by a temperature dependence in terms of the lowest Matsubara frequency.

The microscopic spectral density for the chiral gaussian unitary ensemble was known before the present work, but its behaviour under non-invariant deformations was not. The motivation for this temperature dependent model comes from the chiral phase transition, whose order parameter is given by the microscopic part of the spectral density as discussed in the previous chapter. The model predicts a second order phase transition in the temperature axis with mean field critical exponents. Above the critical temperature the microscopic spectral density vanishes identically and the overall shape of the spectral density separates in to two distinct bulges at high enough temperatures. These predictions about the phase structure of QCD can only be considered as qualitative. More interestingly, we will see that below the critical temperature the functional shape of the microscopic spectral density remains invariant except for a trivial temperature dependent overall scaling of the function.

As with any other random matrix model, the partition function of this model involves an integral over an ensemble of large matrices. As explained in chapter 3 this integral can be separated into a Haar integral over the diagonalizing group and another integral over the eigenvalues of the matrix (with the introduction of a jacobian given by the Vandermonde determinant of the eigenvalues). If the partition function is invariant under unitary transformations the group integrals factor out trivially and the orthogonal polynomial method discussed earlier can be used to analyze the remaining eigenvalue integrals. However the introduction of the temperature dependence ruins the unitary invariance of the random matrix model partition function, and the group integrals are not trivial. Therefore the orthogonal polynomial method is unavailable anymore. Instead we use the supersymmetry method introduced in chapter 2. Using a Hubbard-Stratonovitch transformation we rewrite the partition function in terms of small sized supermatrices. The size of the corresponding supermatrix is fixed and independent of N, the original block size. Therefore the remaining integrals leading to the spectral density for arbitrary temperature can be performed analytically by integrating out the Grassmann variables by brute force. As it will be seen below this method is too cumbersome for the computation of the higher correlation functions. This will require the development of integration techniques over superunitary groups as discussed in sections 4.4 through 4.6.

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In this paper we study a model which was introduced in [127, 221]. This

model is a random matrix model which possesses the chiral and flavor structure of the QCD Dirac operator and a schematic temperature dependence corresponding to the lowest Matsubara frequency. Otherwise, all matrix elements of the Dirac operator are completely random. The temperature dependence is such that this model has a second-order phase transition with mean field critical exponents. Below the critical temperature, chiral symmetry is broken spontaneously; above the critical temperature, it is restored. According to the Banks-Casher formula [22], the order parameter is the spectral density at eigenvalue zero. Physical motivation for this model comes from two rather different directions. First, at zero temperature, it satisfies all Leutwyler-Smilga sum-rules [141], which are identities for chiral QCD in a finite volume. Second, Kocić and Kogut have recently suggested [139] that the chiral phase transition in fermionic systems is driven towards a mean field description because of the fact that the lowest Matsubara frequency is non-zero.

The chiral structure of the Dirac operator forces all eigenvalues to appear in pairs $\pm\lambda$. The spectrum is symmetric about zero. As we shall soon see, it is useful to introduce the microscopic limit of the spectral density which probes the spectrum around zero on a scale set by the distance between adjacent eigenvalues:

$$\rho_S(u) = \lim_{N\to\infty} \frac{1}{N\Xi}\rho(\frac{u}{N\Xi}) \quad . \tag{4.11}$$

Here, Ξ is the temperature-dependent chiral condensate which, according to the Banks-Casher formula [22], is given by

$$\Xi = \frac{\pi\rho(0)}{N} \quad , \tag{4.12}$$

and N is the total number of eigenvalues.

The zero-temperature version of this model has been studied extensively in the literature [86, 49, 196, 152, 153, 154, 42]. It is known as the Laguerre ensemble or the chiral Gaussian Unitary Ensemble (chGUE). Two types of universal behaviour are known to exist in chiral random matrix theories: Spectral correlations in the bulk of the spectrum are universal; the microscopic limit of the spectral density, just introduced, is universal. It has been shown [44, 106, 25, 153, 154] that the chiral structure of the random matrix ensemble does not affect eigenvalue correlations in the bulk of the spectrum. Such level correlations have been observed both experimentally and numerically in many systems [167, 114, 65, 79, 37, 36, 176, 175, 113]. Further, analytic arguments have been presented [33, 16] in favor of the universality of correlations in the bulk of the spectrum of classically chaotic systems.

In [180, 217] we conjectured that the microscopic limit of the spectral density is universal as well. The first argument in support of this conjecture came from instanton liquid calculations [203], where we were able to generate ensembles large enough to permit the calculation of the spectral density in the microscopic limit. A slightly less direct argument came from lattice QCD calculations of the dependence of the chiral condensate on the valence quark mass [51, 199]. Another hint came from the work of the MIT group [15], who studied the chGUE using the supersymmetric method. They found that the microscopic limit of the spectral density is determined by a saddle-point manifold associated with the spontaneous breaking of a symmetry. The first convincing analytical arguments in favor of this conjecture came from recent work by Brézin, Hikami and Zee [42]. They considered families of random matrix models all possessing the chiral structure of the Dirac operator. They discovered the same microscopic limit in all the models they investigated.

In this paper, we offer further evidence in support of the universality of the microscopic limit of the spectral density. Specifically, we investigate the effect of temperature in the chiral random matrix model introduced in [127, 221]. This model differs structurally from the models in [42]. In the random matrix models considered in [42], the unitary symmetry of the probability distribution leaves the spectrum of each element of the ensemble invariant. This invariance is not realized for temperatures $T \neq 0$, and analytic proofs are consequently somewhat more difficult. Using the supersymmetric method of random matrix theory, we obtain an exact expression for the spectral density which is valid for any dimension, n, of the matrices. This enables us to take the microscopic limit. This limit also requires the large-n limit of the spectral density (see (4.12)), which can be evaluated conveniently by means of a saddle-point approximation. In [127], this spectral density was evaluated numerically. It was found to have the well-known semi-circular shape at zero temperature. At high temperature, the shape is given by two disjoint semi-circles with centers located at $\pm\pi T$. In that paper, we also announced the analytic result for the shape of the average spectral density. The result merely requires the solution of a cubic equation. This result has also been obtained by Stephanov [185], who applied an extension of this model to the problem of the relation between the Z_N phase of the theory and the restoration of chiral symmetry.

The organization of this paper is a follows. In the next section we give a definition of the random matrix model and the supersymmetric partition function. In section 4.3.2 the partition function is reduced to a finite-dimensional integral. Symmetries and convergence questions of the partition function are analyzed in section 4.3.3. The exact two-dimensional integral for the resolvent is obtained in section 4.3.4. In section 4.3.5 we derive the large-n limit of

the spectral density and discuss its properties. The microscopic limit of the partition function is evaluated in section 4.3.6, and concluding remarks are made in section 4.3.7. Some notation and conventions as well as a perturbative calculation of the large-n limit of the spectral density are to be found in the appendices presented at the end of this section.

4.3.1 Definition of the random matrix model

In this paper we study the spectrum of the ensemble of matrices

$$H = \begin{pmatrix} 0 & W + \pi T \\ W^\dagger + \pi T & 0 \end{pmatrix} \quad . \tag{4.13}$$

Here, T is the temperature dependence as given by the lowest Matsubara frequency, and W is a complex $n \times n$ matrix distributed according to

$$\exp[-n\Sigma^2 \operatorname{Tr} WW^\dagger] \quad . \tag{4.14}$$

The average spectral density can be expressed as

$$\rho(\lambda) = -\lim_{\epsilon \to 0} \frac{2n}{\pi} \operatorname{Im} G(\lambda + i\epsilon) \quad . \tag{4.15}$$

where the average resolvent $G(z)$,

$$G(z) = \frac{1}{2n} \operatorname{Tr} \overline{\frac{1}{z + i0 - H}} \;=\; -\frac{1}{2n} \frac{\partial \log Z(J)}{\partial J}\bigg|_{J=0} \quad , \tag{4.16}$$

can be obtained from the partition function

$$Z(J) = \int \mathcal{D}W \; \frac{\det(z - H)}{\det(z + J - H)} \; \exp[-n\Sigma^2 \operatorname{Tr} WW^\dagger]. \tag{4.17}$$

The integration measure, $\mathcal{D}W$, is the Haar measure normalized so that $Z(0) = 1$. For a Hermitean matrix, H, the resolvent is analytic in z in the upper complex half-plane. This allows us to calculate the resolvent for purely imaginary z and to perform the analytic continuation to real z at the end of the calculation. As will be seen below, this improves the convergence properties of the integrals in the partition function.

Some properties of this model all already known. At $T = 0$, this model reduces to the well-known Laguerre ensemble. The joint probability distribution of the eigenvalues is known explicitly, as are all correlation functions. In

particular, the average spectral density is a semicircle:

$$\rho(\lambda) = \frac{n\Sigma^2}{\pi}\sqrt{\frac{4}{\Sigma^2} - \lambda^2} \quad . \tag{4.18}$$

We wish to stress that the largest eigenvalue is larger than a typical matrix element by a factor on the order of $\sqrt{n}$.

The temperature dependence of this model was analyzed in [127]. It was shown that, in the thermodynamic limit, this model shows a chiral phase transition at a critical temperature of

$$T_c = \frac{1}{\pi\Sigma} \quad . \tag{4.19}$$

The order parameter is the chiral condensate Ξ with $\Xi \neq$ below T_c and $\Xi = 0$ above T_c. This chiral symmetry is broken spontaneously. For each finite value of n, $\Xi = 0$. A non-zero value of Ξ is obtained only in the thermodynamic limit. Below T_c it was found that in this limit

$$\Xi = \Sigma\ (1 - \pi^2 T^2 \Sigma^2)^{1/2} \quad . \tag{4.20}$$

The Banks-Casher formula (4.12) allows us to convert this value of Ξ into the spectral density $\rho(0)$. In [127], the complete spectral density of this model was determined numerically. At $T = 0$ the result (4.18) was reproduced; at $T = T_c$ we found that $\rho(\lambda) \sim \lambda^{1/3}$. For $T \gg T_c$, the spectral density reduced to two semi-circles centered at $\pm\pi T$ with a radius independent of T. We will present an analytic derivation of these results.

4.3.2 Ensemble average of the partition function

In order to perform the Gaussian integrals, we write the determinant as an integral over the fermionic variables χ and χ^*:

$$\begin{aligned} \det(z - H) \quad = \quad & (2\pi)^{2n} \int \prod_{i=1}^{n} d[\chi_{1i}^*] d[\chi_{1i}] \prod_{i=1}^{n} d[\chi_{2i}^*] d[\chi_{2i}] \\ & \times \exp i \begin{pmatrix} \chi_1^* \\ \chi_2^* \end{pmatrix} \begin{pmatrix} z & -W - \pi T \\ -W^\dagger - \pi T & z \end{pmatrix} \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} \quad . \end{aligned}$$

Similarly, the inverse determinant can be written as an integral over the bosonic variables ϕ and ϕ^*:

$$\begin{aligned} \det{}^{-1}(z - H) \quad = \quad & \tfrac{1}{(2\pi)^{2n}} \int \prod_{i=1}^{n} d[\phi_{1i}] d[{\phi_1}_i^*] \prod_{i=1}^{n} d[\phi_{2i}] d[{\phi_2}_i^*] \\ & \times \exp i \begin{pmatrix} \phi_1^* \\ \phi_2^* \end{pmatrix} \begin{pmatrix} z + J & -W - \pi T \\ -W^\dagger - \pi T & z + J \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \quad . \end{aligned}$$

The conventions for the Gaussian integrals are defined in Appendix A. The factor i in the exponent in (4.21) is chosen so that the integral is convergent for z in the upper complex half-plane. This choice is consistent with the $i\epsilon$ prescription in (4.16). The integral in (4.21) converges independent of the overall phase of the exponent. The present choice of phase ensures that the product of the fermionic and bosonic integrals is one.

The Gaussian integral over W can be performed by completing the squares. The result is a term of fourth order in the integration variables,

$$\exp -\frac{1}{n\Sigma^2}(\chi^*_{2j}\chi_{1i} + \phi^*_{2j}\phi_{1i})(\chi^*_{1i}\chi_{2j} + \phi^*_{1i}\phi_{2j}) \quad . \tag{4.21}$$

We apply the Hubbard-Stratonovitch transformation to each of the terms of fourth order in the bosonic and fermionic variables. Two of the four factors can be decoupled with the help of real integration variables:

$$\begin{aligned}
\exp -\frac{1}{n\Sigma^2}\phi^*_1\cdot\phi_1\,\phi^*_2\cdot\phi_2 &= \int \frac{d\sigma_1 d\sigma_2}{I_b}\exp[-n\Sigma^2(\sigma_1^2+\sigma_2^2) \\
&\quad -(\sigma_1+i\sigma_2)\phi^*_1\cdot\phi_1 + (\sigma_1 - i\sigma_2)\phi^*_2\cdot\phi_2] \ , \\
\exp +\frac{1}{n\Sigma^2}\chi^*_1\cdot\chi_1\,\chi^*_2\cdot\chi_2 &= \int \frac{d\rho_1 d\rho_2}{I_b}\exp[-n\Sigma^2(\rho_1^2+\rho_2^2) \\
&\quad -(\rho_1 - i\rho_2)\chi^*_1\cdot\chi_1 - (\rho_1+i\rho_2)\chi^*_2\cdot\chi_2] \ .
\end{aligned} \tag{4.22}$$

The terms which involve mixed bilinears can be decoupled with the help of Grassmann integrations. We do not encounter convergence problems in the process.

$$\begin{aligned}
\exp -\tfrac{1}{n\Sigma^2}\chi^*_1\cdot\phi_1\,\chi_2\cdot\phi^*_2 &= I_f\int\tfrac{d\alpha^* d\beta}{(i/2)}\exp[-n\Sigma^2\alpha^*\beta + \alpha^*\chi^*_1\cdot\phi_1 - \beta\phi^*_2\cdot\chi_2] \ , \\
\exp \tfrac{1}{n\Sigma^2}\chi_1\cdot\phi^*_1\,\chi^*_2\cdot\phi_2 &= I_f\int\tfrac{d\beta^* d\alpha}{(i/2)}\exp[-n\Sigma^2\beta^*\alpha - \alpha\chi_1\cdot\phi^*_1 + \beta^*\phi_2\cdot\chi^*_2] \quad .
\end{aligned}$$

The constants I_b and I_f are defined such that $I_f/I_b = 1$. (See Appendix A.)

Thus, we obtain the partition function as

$$\begin{aligned}
Z(J) &= \int\prod_{i=1}^n d[\phi_i]\ d[\chi_i]\ d[\sigma]\exp\left[-n\Sigma^2(\sigma_1^2+\sigma_2^2+\rho_1^2+\rho_2^2+\alpha^*\beta+\beta^*\alpha)\right] \\
&\times \exp i\,(\phi^*_1,\phi^*_2,\chi^*_1,\chi^*_2) \\
&\times \begin{pmatrix} z+J+i\sigma_1-\sigma_2 & -\pi T & i\alpha & 0 \\ -\pi T & z+J-i\sigma_1-\sigma_2 & 0 & i\beta \\ i\alpha^* & 0 & z+i\rho_1+\rho_2 & -\pi T \\ 0 & i\beta^* & -\pi T & z+i\rho_1-\rho_2 \end{pmatrix} \\
&\times (\phi_1,\phi_2,\chi_1,\chi_2)^T \ ,
\end{aligned}$$

where

$$d[\sigma] = d\sigma_1 \; d\sigma_2 \; d\rho_1 \; d\rho_2 \; \frac{d\alpha \; d\alpha^* \; d\beta \; d\beta^*}{(i/2)^2} \; . \tag{4.23}$$

If $i\sigma_1$, σ_2, z, and J are all real, the matrix A appearing in the exponent of (4.23) is a graded Hermitean matrix. Then the Gaussian integrals can be performed according to (4.76). This results in

$$Z(J) = \int d[\sigma] \exp\left[-n\Sigma^2(\sigma_1^2 + \sigma_2^2 + \rho_1^2 + \rho_2^2 + \alpha^*\beta + \beta^*\alpha)\right] \; \mathrm{detg}^{-n} A \; , \tag{4.24}$$

where detg$A$ is the graded determinant of A. For a matrix with Grassmann blocks ρ and σ and commuting blocks a and b, it can be shown that

$$\mathrm{detg} \begin{pmatrix} a & \sigma \\ \rho & b \end{pmatrix} = \det{}^{-1} b \; \det(a - \sigma b^{-1} \rho) \; . \tag{4.25}$$

In our case, all blocks a and b are 2×2 matrices, which permits us to evaluate all expressions directly.

We note, however, that the result (4.24) was obtained by interchanging the ϕ_i and σ_i integrations in (4.23). This is allowed only if the ϕ integral is uniformly convergent in σ. Unfortunately, this is not the case when the σ_1 and σ_2 integration paths are along the real axis. This problem can be circumvented by a suitable deformation of the integration paths. Previous studies of random matrix theories within the framework of the sigma model formulation of the Anderson model [173, 173, 169] have stressed the importance of deforming the integration contours in a manner which is consistent with the symmetries of the problem. The same is true for the present problem. In order to interchange the ϕ and σ integrals in (4.23), we must deform the integration contour so that the ϕ integration is uniformly convergent in σ. In order to motivate our choice of contour, we must first consider the symmetries of the partition function.

4.3.3 Symmetries

We wish to study the partition function in the microscopic limit, *i.e.*, the limit $n \to \infty$ with zn held constant. For $z = 0$ (and $J = 0$), the partition function has an additional symmetry. For all temperatures, the bosonic part of the partition function is invariant under the *non-compact* symmetry operation

$$\begin{aligned} \phi_1 &\to e^{+t} \phi_1 \; , \qquad & \phi_1^* &\to e^{+t} \phi_1^* \; , \\ \phi_2 &\to e^{-t} \phi_2 \; , \qquad & \phi_2^* &\to e^{-t} \phi_2^* \; . \end{aligned} \tag{4.26}$$

This induces a hyperbolic rotation of the variables σ_1 and σ_2 of the preceding section,

$$\begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix} \to \begin{pmatrix} \cosh t & i \sinh t \\ -i \sinh t & \cosh t \end{pmatrix} \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix} , \tag{4.27}$$

which clearly reveals the $O(1,1)$ nature of the transformation.

The fermionic part of the partition function is invariant under

$$\begin{aligned} \chi_1 \to e^{+iu} \chi_1 , \qquad & \chi_1^* \to e^{+iu} \chi_1^* , \\ \chi_2 \to e^{-iu} \chi_2 , \qquad & \chi_2^* \to e^{-iu} \chi_2^* , \end{aligned} \tag{4.28}$$

where, *a priori*, u can be either real or complex. In terms of the ρ variables of (4.22), this induces the transformation

$$\begin{pmatrix} \rho_1 \\ \rho_2 \end{pmatrix} \to \begin{pmatrix} \cos u & \sin u \\ -\sin u & \cos u \end{pmatrix} \begin{pmatrix} \rho_1 \\ \rho_2 \end{pmatrix} . \tag{4.29}$$

Because the integration over the Grassmann variables is finite, the volume of the symmetry group must be finite as well. Therefore, u must be real with $u \in [0, 2\pi]$. In other words, the symmetry group is $O(2)$.

The terms of a mixed fermionic-bosonic nature are also affected by the transformations (4.26) and (4.29). This induces a transformation of Grassmann variables introduced through the Hubbard-Stratonovitch transformation.

As is known from studies of random matrix theories, the parametrization of the variables (σ_1, σ_2) and (ρ_1, ρ_2) is dictated by the above symmetries. It is natural and convenient to choose integration variables which lie along and perpendicular to the invariant manifold. In the microscopic limit, integrations along this manifold must be performed exactly, whereas the perpendicular integrations can be performed by saddle-point methods in the limit $n \to \infty$. For the σ variables, we thus choose the parametrization

$$\begin{aligned} \sigma_1 &= -i(\sigma - i\epsilon) \sinh s/\Sigma , \\ \sigma_2 &= (z - i\epsilon) + J + (\sigma - i\epsilon) \cosh s/\Sigma , \end{aligned}$$

where $\sigma \in [-\infty; +\infty]$ and $s \in [-\infty; +\infty]$. After the ϕ_i integration, the $i\epsilon$ appears only in the combination $\sigma - i\epsilon$. Below, we will not write the $i\epsilon$ term explicitly, but it is always understood that it is included in the variable σ. This parametrization renders the ϕ_1 and ϕ_2 integrations uniformly convergent in σ_1 without jeopardizing the convergence of the σ and s integrations. This allows us to interchange the ϕ_i and σ_i integrations leading to the final result

(4.24) of the last section. The term $\sigma_1^2 + \sigma_2^2$ appearing in the first exponent in (4.24) becomes $\sigma^2 + (z+J)^2 + 2(z+J)\sigma\cosh s$ in the parametrization (4.30). It is clear that the integral over s can be convergent only when $z+J$ is purely imaginary. (Recall that σ contains the term $-i\epsilon$.) The transformation (4.26) for $z + J = 0$ reduces to a translation of s and leaves σ invariant.

The ρ variables are also parametrized along and perpendicular to the saddle-point manifold according to

$$\begin{aligned} \rho_1 &= iz + \rho\cos\varphi/\Sigma \ , \\ \rho_2 &= \quad \rho\sin\varphi/\Sigma \ . \end{aligned}$$

The rotation (4.29) leads to a translation of the angle φ and leaves ρ invariant.

Our strategy in dealing with (4.24) is to perform the Grassmann integrations first, *i.e.*, to collect the coefficient of $\alpha\alpha^*\beta\beta^*$. This leaves us with a four-dimensional integral which is the exact analytical result for the partition function for any finite n. In the thermodynamic limit and for $z \sim \mathcal{O}(1)$, the remaining integrations can be performed with a saddle-point approximation. This result is obtained in section 6. In the microscopic limit, z will be $\mathcal{O}(1/n)$, and the integration over the invariant manifold must be performed exactly. The radial integrals can be approximated to leading order in $1/n$. (See section 5.)

The $T = 0$ problem has been investigated previously using the supersymmetric method [15, 42]. In [15], the saddle-point manifold was constructed in a manner similar to that used for the problem of invariant random matrix ensembles. (In this regard, see [76, 216].) In [42], the convergence difficulties were circumvented in an elegant fashion by the use of spherical coordinates for the variables ϕ_1 and ϕ_2. Unfortunately, a direct generalization of this approach is not possible for the present case of non-zero temperatures. When $T \neq 0$, the angles between the complex vectors ϕ_1 and ϕ_2 also enter in the integration variables.

4.3.4 Exact result for the spectral density at finite n

Because the fermionic blocks of the matrix A are nilpotent, the right side of (4.25) can be expanded in a finite number of terms. The n-th power of the inverse of the graded determinant can be written as

$$\mathrm{detg}^{-n}\begin{pmatrix} a & \sigma \\ \rho & b \end{pmatrix} = \left(\frac{\det b}{\det a}\right)^n \times$$

$$\times \quad \left(1 + n\mathrm{Tr}a^{-1}\sigma b^{-1}\rho + \frac{n}{2}\mathrm{Tr}(a^{-1}\sigma b^{-1}\rho)^2 + \frac{n^2}{2}\mathrm{Tr}^2 a^{-1}\sigma b^{-1}\rho\right). \tag{4.30}$$

Terms in the partition function which are of fourth order in the Grassmann variables can be obtained by supplementing the above terms with factors $\alpha^*\beta$ and $\beta^*\alpha$ from the exponent in (4.24). The result is

$$\begin{aligned} Z(J) &= \tfrac{n^2\Sigma^4}{\pi^2}\int d\sigma_1 d\sigma_2 d\rho_1 d\rho_2 \left[(1-\tfrac{\pi^2T^2}{D\Delta\Sigma^2})^2 - \tfrac{(D+\pi^2T^2)(\Delta+\pi^2T^2)+\pi^2T^2(\Delta-D)/n}{D^2\Delta^2\Sigma^4}\right] \\ &\times \qquad \left(\tfrac{\Delta}{D}\right)^n \exp[-n\Sigma^2(\sigma_1^2+\sigma_2^2+\rho_1^2+\rho_2^2)]\ , \end{aligned}$$

where D is the determinant of the boson-boson block,

$$D = (z + J + i\sigma_1 - \sigma_2)(z + J - i\sigma_1 - \sigma_2) - \pi^2T^2\ , \tag{4.31}$$

and Δ is the determinant of the fermion-fermion block,

$$\Delta = (z + i\rho_1 + \rho_2)(z + i\rho_1 - \rho_2) - \pi^2T^2\ . \tag{4.32}$$

In (4.31) the variables σ_i and ρ_i are parametrized according to (4.30) and (4.30). temperature by

$$t = \pi T\Sigma \tag{4.33}$$

the resulting form of $Z(J)$ simplifies to

$$\begin{aligned} Z(J) &= \frac{-in^2}{\pi^2}\int_{-\infty}^{\infty}\sigma d\sigma\int_{-\infty}^{\infty}ds\int_0^{\infty}\rho d\rho\int_0^{2\pi}d\varphi \\ &\times \left[(1-\frac{t^2}{(t^2-\sigma^2)(\rho^2+t^2)})^2 + \frac{\sigma^2\rho^2+t^2(\sigma^2+\rho^2)/n}{(\sigma^2-t^2)^2(\rho^2+t^2)^2}\right] \\ &\times \left(\frac{\rho^2+t^2}{t^2-\sigma^2}\right)^n e^{-n(\sigma^2+\rho^2+2(z+J)\Sigma\sigma\cosh s+2iz\Sigma\rho\cos\varphi+\Sigma^2((z+J)^2-z^2))} \end{aligned} \tag{4.34}$$

The integrations over s and φ can be expressed in terms of Bessel functions. Differentiation of the partition function with respect to J at $J = 0$ gives us the resolvent which we desire. Thus, the final result of this section is

$$\begin{aligned} G(z) &= \frac{2in}{\pi}\int\sigma d\sigma\rho d\rho \\ &\times \left[(1-\frac{t^2}{(t^2-\sigma^2)(\rho^2+t^2)})^2 + \frac{\sigma^2\rho^2+t^2(\sigma^2+\rho^2)/n}{(\sigma^2-t^2)^2(\rho^2+t^2)^2}\right] \\ &\times \left(\frac{\rho^2+t^2}{t^2-\sigma^2}\right)^n (2z\Sigma^2K_0(2n\Sigma z\sigma) - 2n\Sigma\sigma K_1(2n\Sigma z\sigma)) \\ &\times J_0(2n\Sigma z\rho)\exp[-n(\sigma^2+\rho^2)]. \end{aligned} \tag{4.35}$$

Again, we remind the reader that σ contains a term $-i\epsilon$. The integral over σ can thus be performed by successive partial integrations. Details regarding the calculation of this kind of integral can be found in [95]. This result has been obtained for z purely imaginary. Since the modified Bessel functions have a cut for $z\sigma < 0$, we can analytically continue this expression anywhere in the upper half-plane.

4.3.5 The large-n limit of the average spectral density

For $n \to \infty$ and $z \sim \mathcal{O}(1)$, all integrals in (4.34) can be performed by a saddle-point approximation. Because we started with a supersymmetric partition function, the Gaussian fluctuations about the saddle point give an overall constant of unity, *i.e.*, $Z(0) = 1$ in (4.34). Using the relation (4.16) to determine the resolvent from the partition function (4.34), we find that

$$G(z) = \Sigma(\Sigma z + \bar{\sigma}\cosh\bar{s}) \ , \tag{4.36}$$

where $\bar{\sigma}$ and $\bar{s}$ are the saddle-point values of these variables. The saddle-point equation for s is trivial with solution $\bar{s} = 0$. The equation for $\bar{\sigma}$ is more complicated

$$\frac{\bar{\sigma}}{\bar{\sigma}^2 - t^2} - (\bar{\sigma} + \Sigma z) = 0 \ . \tag{4.37}$$

This equation can be rewritten as an equation for the ensemble averaged resolvent

$$G^3/\Sigma^4 - 2zG^2/\Sigma^2 + G(z^2 - \pi^2 T^2 + 1/\Sigma^2) - z = 0 \ . \tag{4.38}$$

At $T = 0$, this equation reduces to

$$(G - z)(G^2/\Sigma^2 - zG + 1) = 0 \tag{4.39}$$

with a non-trivial solution

$$G(z) = \Sigma^2 \ \frac{z \pm i(4/\Sigma^2 - z^2)^{1/2}}{2} \ . \tag{4.40}$$

As indicated in (4.15) above, the associated spectral density is simply the imaginary part of the branch of $G(z)$ with the negative sign,

$$\rho(\lambda) = \frac{n\Sigma^2}{\pi}(4/\Sigma^2 - \lambda^2)^{1/2} \ , \tag{4.41}$$

which is the familiar semicircle normalized to the total number of eigenvalues.

For $z = 0$, the saddle-point equation (4.38) simplifies to

$$G^3 + \Sigma^4 G(1/\Sigma^2 - \pi^2 T^2) = 0 \tag{4.42}$$

with the solution

$$G(0) = -i\Sigma\sqrt{1 - \pi^2 T^2 \Sigma^2} \ . \tag{4.43}$$

This corresponds to the spectral density

$$\rho(0) = \frac{2n\Sigma}{\pi}\sqrt{1 - \pi^2 T^2 \Sigma^2} \quad . \tag{4.44}$$

Using the Banks-Casher formula (4.12), we immediately obtain the chiral condensate (4.20) in agreement with [127].

In order to determine the high-temperature limit of the spectral density, it is most convenient to return to the saddle-point equation (4.37). It is clear that, for $z \approx \pi T$, this equation can only be satisfied for $\bar{\sigma} \approx -t$. In the partition function (4.34) we can approximate the logarithmic term

$$\log(t^2 - \sigma^2) \approx \log(2t) + \log(t + \sigma) \quad . \tag{4.45}$$

This leads us to the high-temperature limit of the saddle-point equation

$$-\frac{1}{\bar{\sigma} + t} - (\bar{\sigma} + \Sigma z) = 0 \ . \tag{4.46}$$

The solution for the resolvent is

$$G(z) = \frac{\Sigma}{2}(\Sigma z - t - i\sqrt{2 - (\Sigma z - t)^2}) \quad . \tag{4.47}$$

This results in a semicircular spectral density of radius $\sqrt{2}$ located at $z = \pi T$. An identical argument leads to another semicircular contribution to the spectral density of radius $\sqrt{2}$ centered at $z = -\pi T$. Of course, we can arrive at the same conclusion working directly from (4.38). For $z \approx \pi T$, the resolvent $G(z) \sim \mathcal{O}(1)$, and the first term in (4.38) will be sub-leading in the high-temperature limit. This leads immediately to (4.47).

Finally, we consider the case at the critical temperature, $T = \Sigma/\pi$, with z in the neighborhood of 0. Then, the saddle-point equation for G reduces to

$$G^3 = z \tag{4.48}$$

with solutions $(z\Sigma^4)^{1/3}$, $(z\Sigma^4)^{1/3}\exp(\pi i/3)$ and $(z\Sigma^4)^{1/3}\exp(2\pi i/3)$. Only the last of these gives rise to a positive definite spectral density with

$$\rho(\lambda) = \frac{n\Sigma\sqrt{3}}{\pi}(\lambda\Sigma)^{1/3} \tag{4.49}$$

in agreement with the mean field critical exponent of $\delta = 3$ for this model.

The equation for the resolvent (4.38) also enables us to obtain a simple recursion relation for the moments of the spectral density. Expanding $G(z)$ in terms of these moments,

$$G(z) = \sum_n \frac{M_{2n}}{z^{2n+1}} \ , \tag{4.50}$$

we obtain

$$M_{2n+2} = (T^2 - \frac{1}{\Sigma^2})M_{2n} + \frac{2}{\Sigma^2}\sum_{k+l=n} M_{2k}M_{2l} - \frac{1}{\Sigma^4}\sum_{k+l+m=n-1} M_{2k}M_{2l}M_{2m} \tag{4.51}$$

The evident initial condition, $M_0 = 1$, immediately leads us to

$$M_2 = T^2 + y^2, \qquad M_4 = T^4 + 4y^2T^2 + 2y^4,$$

et cetera. Without too much effort, it is possible to use standard combinatoric methods to find the general result for the $(2n)$-th moment,

$$M_{2n} = \sum_{k=0}^{n} y^{2k}T^{2(n-k)}\frac{1}{k+1}\left(\begin{array}{c} n \\ k \end{array}\right)\left(\begin{array}{c} 2n \\ k \end{array}\right) \ . \tag{4.52}$$

4.3.6 The microscopic limit of the partition function

The 'microscopic limit' denotes the investigation of the spectral density in the vicinity of $z = 0$ on a scale set by the average level spacing. More precisely, we take the limit $n \to \infty$ while keeping nz fixed, as indicated in (4.11). We start from the expression (4.24) for the partition function. In the thermodynamic limit, the σ and ρ integrations can be performed by a saddle-point method. The saddle-point equations read

$$\begin{aligned} \frac{\rho}{\rho^2 + t^2} - \rho &= 0 \ , \\ \frac{\sigma}{t^2 - \sigma^2} - \sigma &= 0 \ , \end{aligned} \tag{4.53}$$

with solutions

$$\begin{aligned} \bar{\rho}^2 &= 1 - t^2 \ , \\ \bar{\sigma}^2 &= t^2 - 1 \ . \end{aligned} \tag{4.54}$$

For temperatures less than the critical temperature, $\bar{\rho}$ is real and $\bar{\sigma}$ is purely imaginary. The integration range of ρ is the positive real axis. Therefore, the sign of $\bar{\rho}$ is positive. The σ integration ranges from $-\infty$ to $+\infty$. In order the reach the σ saddle point, we must deform the integration contour. Because of the modified Bessel functions which appear in our expression (4.35) for the resolvent, there is a cut in the complex σ-plane for σz on the negative real axis. The cut of the modified Bessel function is then $i\epsilon$ above the positive real axis for negative z and $i\epsilon$ above the negative real axis for positive z. Therefore, independent of the sign of z, only the saddle point with a negative imaginary part can be reached by a deformation of the contour. Thus,

$$\bar{\sigma} = -i \left(1 - t^2\right)^{1/2} , \tag{4.55}$$

for $t < 1$.

At the saddle point, the pre-exponential factor vanishes:

$$\left(1 - \frac{t^2}{(t^2 - \bar{\sigma}^2)(\bar{\rho}^2 + t^2)}\right)^2 + \frac{\bar{\sigma}^2\bar{\rho}^2 + t^2(\bar{\sigma}^2 + \bar{\rho}^2)/n}{(t^2 - \bar{\sigma}^2)^2(\bar{\rho}^2 + t^2)^2} = 0 \ . \tag{4.56}$$

Given (4.54), it is trivial that this equation is satisfied when $t = 0$. However, the vanishing of this pre-exponential factor for arbitrary t is remarkable and unexpected. This fact is responsible for the 'universal' behaviour of the microscopic limit of the spectral density. As a consequence, the $\mathcal{O}(1/n)$ term in this factor does not contribute to the resolvent to leading order in $1/n$. The zK_0 term in the pre-exponent is also of subleading order ($z \sim \mathcal{O}(1/n)$). This leads to the following result for the resolvent in the microscopic limit:

$$\begin{aligned} G(z) \ &= \ -\frac{4n^2 i\Sigma}{\pi} \int \sigma d\sigma \rho d\rho \\ &\times \ \left[\left(1 - \frac{t^2}{(t^2 - \sigma^2)(\rho^2 + t^2)}\right)^2 + \frac{\sigma^2\rho^2}{(\sigma^2 - t^2)^2(\rho^2 + t^2)^2}\right] \\ &\times \ \sigma K_1(2n\Sigma z\sigma)\, J_0(2n\Sigma z\rho) \\ &\times \ \exp[-n(\sigma^2 + \rho^2 + \log(t^2 - \sigma^2) - \log(\rho^2 + t^2))] \ . \end{aligned} \tag{4.57}$$

In order to proceed, we make the substitution

$$\begin{aligned} \sigma &= \bar{\sigma} + \delta\sigma \ , \\ \rho &= \bar{\rho} + \delta\rho \end{aligned} \tag{4.58}$$

in (4.57) and keep only those terms which contribute to leading order in $1/n$, *i.e.*, terms through second order in $\delta\rho$ and $\delta\sigma$. The exponent in (4.57) then becomes

$$\exp(2n\bar{\sigma}^2\delta\sigma^2 - 2n\bar{\rho}^2\delta\rho^2) \ . \tag{4.59}$$

The product of the terms in square brackets in (4.57) and $\sigma^2\rho$ can be expanded as

$$\begin{aligned} &\bar{\sigma}^2\bar{\rho}[2\bar{\rho}^3\delta\rho + 2\bar{\sigma}^3\delta\sigma + \bar{\rho}^2(-1+8t^2)\delta\rho^2 \\ &+\bar{\sigma}^2(+1+8t^2)\delta\sigma^2 - \bar{\sigma}\bar{\rho}(2+8t^2)\delta\rho\delta\sigma] \ . \end{aligned} \tag{4.60}$$

It is clear already at this point that all temperature dependence enters through the scale factors $\bar{\sigma}$ and $\bar{\rho}$. The terms of $\mathcal{O}(\delta\rho\delta\sigma)$ vanish upon integration with the exponential factor (4.59). Since $\langle\delta\rho^2\rangle = \langle\delta\sigma^2\rangle$ and $\bar{\sigma}^2 = -\bar{\rho}^2$, the other terms involving $8t^2$ cancel as well. To complete the calculation, we need only expand the Bessel functions to first order

$$K_1(2n\Sigma z\sigma)\, J_0(2n\Sigma z\rho) = K_1 J_0 + 2n\Sigma z\delta\sigma\, K_1'\, J_0 + 2n\Sigma z\delta\rho\, K_1\, J_0'. \tag{4.61}$$

where the Bessel functions K_1 and J_0 and their derivatives appearing on the right of this equation are to be evaluated at their saddle points which are $2n\Sigma z\bar{\sigma}$ and $2n\Sigma z\bar{\rho}$, respectively. In order to arrive at the final result, we form the product of this expression and (4.60), collect the coefficients of $\delta\rho^2$ and $\delta\sigma^2$, and perform the Gaussian integrations over $\delta\rho$ and $\delta\sigma$ according to

$$\begin{aligned} \langle\delta\rho^2\rangle &= \frac{1}{2}\frac{\sqrt{\pi}}{(2n\bar{\rho}^2)^{3/2}}\frac{\sqrt{\pi}}{(-2n\bar{\sigma}^2)^{1/2}} \ , \\ \langle\delta\sigma^2\rangle &= \frac{1}{2}\frac{\sqrt{\pi}}{(2n\bar{\rho}^2)^{1/2}}\frac{\sqrt{\pi}}{(-2n\bar{\sigma}^2)^{3/2}} \ . \end{aligned} \tag{4.62}$$

The result is

$$G(z) = -\frac{i\Sigma}{2}\frac{\bar{\sigma}^2}{\bar{\rho}^3}\left[K_1J_0(\bar{\sigma}^2 - \bar{\rho}^2) + 4nz\bar{\rho}^3\Sigma K_1J_0' + 4nz\bar{\sigma}^3\Sigma K_1'J_0\right] \ . \tag{4.63}$$

If we make use of the identities

$$\begin{aligned} J_0' &= -J_1 \ , \\ K_1'(z) &= -K_0(z) - \frac{1}{z}K_1(z) \ , \end{aligned} \tag{4.64}$$

we discover that the terms proportional to $K_1 J_0$ cancel. This leaves us with

$$G(z) = i2nz\Sigma^2(1-t^2)(K_1 J_1 + iK_0 J_0) \ . \tag{4.65}$$

Finally, we can explicitly separate the resolvent into its real and imaginary parts by using two more elementary identities:

$$\begin{aligned} K_1(-iz) &= -\frac{\pi}{2}[J_1(z) + iN_1(z)] \ , \\ K_0(-iz) &= \frac{\pi}{2} i[J_0(z) + iN_0(z)] \ . \end{aligned} \tag{4.66}$$

The final result for the microscopic spectral density is thus

$$\rho(\lambda) = 2n^2\lambda\Sigma^2(1-t^2)(J_1^2(2n\lambda\Sigma\sqrt{1-t^2}) + J_0^2(2n\lambda\Sigma\sqrt{1-t^2})). \tag{4.67}$$

As noted above, the microscopic limit of this model has previously been considered for the special case $t = 0$ [180, 217, 203, 15, 42]. Our result is in agreement with this earlier work. Now, however, we can also consider the microscopic limit for general $t \neq 0$. At finite temperature, the temperature enters *only* through the temperature-dependent modification of the chiral condensate which was obtained in [127]. As defined in (4.11) with $\rho(0)$ given by (4.44), the microscopic limit is strictly independent of the temperature:

$$\rho_S(u) = \frac{u}{2}[J_0^2(u) + J_1^2(u)] \ . \tag{4.68}$$

4.3.7 Conclusions

In this paper, we have studied a random matrix model with the chiral structure of the QCD Dirac operator and a temperature dependence characteristic of the lowest Matsubara frequency. This model possesses the global color and flavor symmetries of QCD. It undergoes a chiral phase transition with critical exponents given by mean field theory.

Using the supersymmetric method for random matrix theories, we have found an exact, analytic expression for the average spectral density of this model. The result has the form of a two-dimensional integral which is valid for matrices of any dimension. In the large-n limit, these integrals can be performed using a saddle-point approximation. The spectral density then follows from the solution of an elementary cubic equation and nicely confirms our earlier numerical work [127].

Our primary result is that the spectral density in the microscopic limit is strictly independent of the temperature below the critical temperature of this

model. This result supports the recent work of Brézin, Hikami and Zee, who investigated several families of random matrix models and found the same microscopic limit of the spectral density in all cases. As noted in the introduction, our model differs from the models considered by these authors in an essential way. In each of their models, the spectrum of each element in the ensemble is strictly invariant under the unitary symmetry of the probability distribution. In the present model this symmetry is violated for $T \neq 0$. Thus, agreement between the microscopic limit of the spectral density for our model and the models of Brézin, Hikami and Zee increases our confidence in the universality of this quantity.

In lattice QCD simulations the microscopic limit of the spectral density enters in the valence quark mass dependence of the chiral condensate. This quantity has been calculated for a variety of temperatures [51], and it has been shown that the results below the critical temperature and not too large valence quark masses fall on a universal curve that can be obtained from the microscopic limit of the spectral density [199]. The present work provides a proper theoretical foundation of this analysis.

In our derivations, the symmetries of the partition function played a crucial role. The universal behavior was closely related to the existence of an invariant saddle-point manifold generated by these symmetries. This suggests that the 'miraculous' cancellation of the temperature dependence of the microscopic spectral density found here is not a coincidence. It would be very interesting to obtain this result using more general arguments. Recent work by Guhr [98] on the superposition of two matrix ensembles appears to offer a promising method towards this goal.

4.3.8 Appendix A: Notations and conventions

In this appendix we summarize our notations and conventions. For a more detailed discussion regarding the motivation for these conventions, we refer to [76, 216].

The integration measure for complex Gaussian integrals is defined such that

$$\int \frac{d\phi^* \; d\phi}{2\pi} \exp +i\phi^*\phi = 1 \; . \tag{4.69}$$

For Grassmann integrals, the measure is defined so that

$$2\pi \int d\chi^* \; d\chi \exp +i\chi^*\chi = 1 \; . \tag{4.70}$$

A graded vector or supervector is defined by

$$\Phi = \begin{pmatrix} \phi \\ \chi \end{pmatrix} , \tag{4.71}$$

with ϕ a commuting vector of length n and χ an anti-commuting vector of dimension m. The corresponding supermatrix which acts on this vector has the structure

$$A = \begin{pmatrix} a & \sigma \\ \rho & b \end{pmatrix} , \tag{4.72}$$

where a and b are complex matrices of dimension $n \times n$ and $m \times m$, respectively. The entries in the $n \times m$ dimensional matrix σ and the $m \times n$ dimensional matrix ρ are Grassmann variables. The graded trace of the matrix A is defined as

$$\mathrm{Trg} A = \mathrm{Tr} a - \mathrm{Tr} b \ . \tag{4.73}$$

The Hermitean conjugate of A is defined as

$$A^\dagger = \begin{pmatrix} a^\dagger & \rho^\dagger \\ -\sigma^\dagger & b^\dagger \end{pmatrix} , \tag{4.74}$$

where the $\dagger$ denotes transposition and complex conjugation. A graded matrix is called Hermitean if $A^\dagger = A$. We use complex conjugation of the second kind for Grassmann variables, *i.e.*, $\chi^{**} = -\chi$. The graded determinant is defined as

$$\mathrm{detg}\ A = \exp(\mathrm{Trg} \log A) \ . \tag{4.75}$$

With this definition, we obtain the following natural result for a Hermitean, graded matrix:

$$\int \prod_{i=1}^{n} d[\phi_i^*]\ d[\phi_i]\ d[\chi_i^*]\ d[\chi_i]\ \ \exp + i \Phi^* A \Phi = \frac{1}{\mathrm{detg}\, A}. \tag{4.76}$$

4.3.9 Appendix B: Perturbative evaluation of the average spectral density

In this appendix, we derive the large-n limit of the resolvent without employing the supersymmetric method. Because the operator (4.13) has only

a finite support, it is possible to expand the resolvent in a geometric series in $1/(z-K)$ for z sufficiently large. Here, K is the matrix

$$K = \begin{pmatrix} 0 & \pi T \\ \pi T & 0 \end{pmatrix} . \tag{4.77}$$

One finds by inspection that $G(z)$ satisfies

$$G(z) = \mathrm{Tr}\frac{1}{z-K} + \mathrm{Tr}\frac{1}{z-K}\overline{\begin{pmatrix} 0 & W \\ W^\dagger & 0 \end{pmatrix}\mathcal{G}\begin{pmatrix} 0 & W \\ W^\dagger & 0 \end{pmatrix}\mathcal{G}} \tag{4.78}$$

where $\mathcal{G}$ is the matrix

$$\mathcal{G} = \overline{\frac{1}{z-H}} , \tag{4.79}$$

and the bar denotes averaging over the probability distribution (4.14). It should be clear that $\mathcal{G}$ is block diagonal with the block structure

$$\mathcal{G} = \begin{pmatrix} g\mathbf{1}_n & h\mathbf{1}_n \\ h\mathbf{1}_n & g\mathbf{1}_n \end{pmatrix}, \tag{4.80}$$

where $\mathbf{1_n}$ is the $n \times n$ identity matrix. Therefore, we find that $G(z) = g$. The average over W can be carried out immediately to give

$$\overline{\begin{pmatrix} 0 & W \\ W^\dagger & 0 \end{pmatrix}\mathcal{G}\begin{pmatrix} 0 & W \\ W^\dagger & 0 \end{pmatrix}} = \frac{1}{n\Sigma^2}\begin{pmatrix} g\mathbf{1}_n & 0 \\ 0 & g\mathbf{1}_n \end{pmatrix} . \tag{4.81}$$

This yields the following matrix equation for g and h:

$$\begin{pmatrix} z & -\pi T \\ -\pi T & z \end{pmatrix}\begin{pmatrix} g & h \\ h & g \end{pmatrix} = \mathbf{1} + \frac{1}{\Sigma^2}\begin{pmatrix} g & 0 \\ 0 & g \end{pmatrix}\begin{pmatrix} g & h \\ h & g \end{pmatrix} , \tag{4.82}$$

which leads to the two independent equations

$$\begin{aligned} zg - \pi T h &= 1 + \frac{1}{\Sigma^2}g^2 , \\ zh - \pi T g &= \frac{1}{\Sigma^2}gh . \end{aligned} \tag{4.83}$$

Elimination of h yields the equation

$$zg - \frac{\pi^2 T^2 g}{z - g/\Sigma^2} = 1 + \frac{1}{\Sigma^2}g^2 , \tag{4.84}$$

which agrees with (4.37). Evidently, it can be rewritten as a cubic equation for g.

4.4 Interlude: Harish-Chandra - Itzykson - Zuber integrals relevant for chiral ensembles

As discussed before the evaluation of a random matrix model partition function involves an integration over a set of matrices. Usually this can be performed easier if the matrices in question are parametrized with respect to their eigenvalues and their diagonalizing groups. A certain class of group integrals become important in this case. The group integrals relevant for the standard gaussian unitary ensemble had been worked out by Itzykson and Zuber [122] and have the form

$$\int dU \exp\left(i\mathrm{Tr}\, RUSU^\dagger\right) = c\frac{\det[e^{ir_k s_l}]_{kl}}{\Delta(R)\Delta(S)}. \tag{4.85}$$

Here $U \epsilon U(N)$, dU is the Haar measure for $U(N)$ and R and S are diagonal real matrices. Also, c is a normalization constant and $\Delta(X) = \det[x_l^{k-1}]$ is the Vandermonde determinant. As it was later realized, the general form of this formula was discovered earlier by the Indian mathematician Harish-Chandra [115]. Hence such group integrals are referred to as Harish-Chandra-Itzykson-Zuber integrals.

The proof of this formula [122] proceeds by recognizing the left hand side of (4.85) to be a part of the kernel of a heat equation and by explicitly constructing the solutions of this heat equation.

For the study of chiral ensembles a different integration theorem is needed. The partition function in (4.3) involves an integral over the nonzero block of the Dirac operator, $\mathbf{C}$, which is an arbitrary complex matrix - in contrast to the hermitian matrices that appear in the standard gaussian unitary ensemble.

It can be shown [119] that an arbitrary complex matrix can be diagonalized using two unitary matrices:

$$\begin{aligned} \mathbf{C} &= UXV^\dagger, \\ (U,V) &\quad \epsilon \quad (U(N)\times U(N))/U(1)^N. \end{aligned} \tag{4.86}$$

Therefore the group integrals relevant for chiral gaussian unitary ensemble have a form involving two unitary matrices:

$$\int d\mu(U,V) \exp\left(\mathrm{ReTr}\, RUSV^\dagger\right) = c'\frac{\det[I_0(r_k s_l)]_{kl}}{\Delta(R^2)\Delta(S^2)}. \tag{4.87}$$

Here $d\mu(U,V)$ is the Haar measure of $(U(N)\times U(N))/U(1)^N$, I_0 is a Bessel function and R, S and $\Delta(X)$ are as before.

Remarkably the heat kernel method used in the proof of (4.85) can be adapted for a proof of (4.87). This requires the solution of a heat equation in curvilinear coordinates. The proper parametrization to separate the heat equation naturally leads into the Bessel kernel seen in (4.87). The next section contains a detailed proof of this last formula as well as its generalization relevant for rectangular blocks, $\mathbf{C}$.

As an additional remark we would like to point out that in the context of the supersymmetry method of random matrix theory this integral requires a generalization to supermatrices [103, 124]: (Conventions and relevant definitions regarding supermatrices can be found in the appendix.)

$$\int d\mu(U,V) e^{-\mathrm{str}\,[(\sigma-\rho)^\dagger(\sigma-\rho)]} = \frac{1}{(k!)^2} \frac{\det \gamma(s_p^0, r_q^0) \det \gamma(s_m^1, r_n^1)}{B(S^2)B(R^2)} \ . \qquad (4.88)$$

Here σ and ρ are two $(k+k) \times (k+k)$ arbitrary complex super matrices with eigenvalues $(s_p^0; is_p^1)$ and $(r_p^0; ir_p^1)$, respectively. (The superscript 0 or 1 indicates bosonic and fermionic eigenvalues.) The berezinian $B(X)$ is defined in the appendix as well. The quantity $\gamma(s,r)$ is defined as

$$\gamma(s,r) = \exp\left[-s^2 - r^2\right] I_0(2sr) \ . \qquad (4.89)$$

This formula will be of utmost importance in section 4.6, where we analytically evaluate the higher correlations functions of a non-invariant chiral random matrix model.

4.5 Finite volume partition functions and Itzykson-Zuber integrals

In this section we study Itzykson-Zuber type integrals corresponding to arbitrary complex rectangular matrices. The motivation for this method came from an attempt to compute higher correlation functions of chiral ensembles using a method developed by Guhr [95] where he used the standard hermitian Itzykson-Zuber integrals for a study the of the gaussian unitary ensemble.

Originally we have studied the supersymmetric generalization given in (4.88) as this is what is needed to express the higher correlation functions in the supersymmetry method. However it turned out later that Guhr and Wettig have also been working on the same problem unknowingly. After their publication on these Itzykson-Zuber integrals, [103], we have published only what is not covered by their work, namely the case for rectangular block matrices. The author would like to take this opportunity to thank again T. Guhr

and T. Wettig for their cooperation which subsequently led to the simultaneous publication of the study of higher correlation functions in chiral gaussian unitary ensemble, given in section 4.6, which was the natural consequence of (and the motivation for) the work in this section.

Another remarkable property of Itzykson-Zuber integrals of the type developed below is that they allow an exact evaluation of the finite volume (mesoscopic) partition function of QCD for arbitrary quark masses as developed by Leutwyler and Smilga [141]. In their work, where they also derive the Leutwyler-Smilga sum rules, they evaluate this partition function only for identical masses.

This work has been published in *Phys. Lett.* **B387**: 355-360, (1996), which we reproduce here except for minor changes.

In general, numerical simulations offer our only access to the QCD partition function. For sufficiently small volumes, however, the partition function is dominated by the constant modes which makes analytic treatment possible. (See, *e.g.*, ref. [198].) In a similar spirit, Leutwyler and Smilga [141] identified a parameter range within which the mass dependence of the partition function is completely determined by the underlying structure of broken chiral symmetry. This range is given as

$$\frac{1}{\Lambda} \ll L \ll \frac{1}{m_\pi} \ , \tag{4.90}$$

where L is the linear size of the 4-dimensional Euclidean box, Λ is a typical hadronic mass scale and m_π is the mass of the Goldstone modes (*i.e.*, $m_\pi \sim \sqrt{m\Lambda}$ for quark mass m). The lower limit of this range ensures that the partition function is dominated by the Goldstone modes. The upper limit ensures that these modes can be treated as constant modes. If chiral symmetry is broken according to $SU(N_f) \times SU(N_f) \to SU(N_f)$, the QCD partition function in the range (4.90) and for vacuum angle θ is given by [141]

$$Z(M,\theta) = \int_{U\in SU(N_f)} dU \exp\left(V\Sigma\,\mathrm{Re}(e^{i\theta/N_f}\ \mathrm{tr}\ UM)\right) \ . \tag{4.91}$$

Here, Σ is the chiral condensate, and M is the mass matrix which can be taken as diagonal without loss of generality. In the sector with topological charge ν, the partition function is given as

$$Z_\nu(M) = \frac{1}{2\pi}\int_0^{2\pi} d\theta e^{-i\nu\theta} Z(\theta) \ . \tag{4.92}$$

For equal quark masses, Leutwyler and Smilga obtained an exact analytic expression for this partition function

$$Z_\nu(m) = \det_{ij} I_{\nu+j-i}(m\,V\,\Sigma)\ , \tag{4.93}$$

where i and j run from $1, \cdots, N_f$. The importance of this partition function lies in the fact that it enables us to determine the volume dependence of the chiral condensate. This is of interest in lattice QCD simulations, where the volume is necessarily finite.

Recently, lattice QCD calculations have been performed to determine the connected and disconnected contributions to the scalar susceptibility [137]. The determination of these susceptibilities requires differentiation of the partition function with respect to two *different* masses. In order to determine volume dependence of the scalar susceptibility in the range (4.90), we require a generalization of (4.93) to different quark masses. This is the primary objective of the present paper.

In the next section , we analyze the two flavor case and generalize the result to an arbitrary number of flavors. Later, we present a generalization of the Itzykson-Zuber formula valid for arbitrary *rectangular* complex matrices. For *square* complex matrices, our result reduces to that obtained by Wettig and Guhr [103]. Finally, we derive the finite volume partition function for different masses in the sector of zero topological charge. We also make a conjecture of the result for arbitrary topological charge and show that it reduces to (4.93) for the special case of equal quark masses.

4.5.1 The finite volume partition function for $N_f = 2$ and its generalization to arbitrary N_f.

For two flavors and in the range (4.90), the QCD-partition function is known for different quark masses. For vacuum angle θ, it is given as

$$Z(\theta) = \frac{2}{V\,\Sigma\,\mu} I_1(V\,\Sigma\,\mu)\ , \tag{4.94}$$

where I_1 is a modified Bessel function and where

$$\mu^2 = m_1^2 + m_2^2 + 2m_1 m_2 \cos(\theta)\ . \tag{4.95}$$

The partition function in the sector with topological charge ν can be obtained by integrating over θ according to (4.92). Remarkably, this integral can be

expressed analytically. After some manipulations the result can be written as

$$Z_\nu(N_f=2) = \frac{2}{x_2^2-x_1^2}\det\begin{vmatrix} I_\nu(x_1) & x_1 I'_\nu(x_1) \\ I_\nu(x_2) & x_2 I'_\nu(x_2) \end{vmatrix} \tag{4.96}$$

where

$$x_k = m_k V\,\Sigma. \tag{4.97}$$

This suggests the generalization to three flavors

$$\begin{aligned} Z_\nu(N_f=3) &= \frac{16}{(x_2^2-x_1^2)(x_2^2-x_3^2)(x_3^2-x_1^2)} \\ &\times \det\begin{vmatrix} I_\nu(x_1) & x_1 I'_\nu(x_1) & x_1^2 I''_\nu(x_1) \\ I_\nu(x_2) & x_2 I'_\nu(x_2) & x_2^2 I''_\nu(x_2) \\ I_\nu(x_3) & x_3 I'_\nu(x_3) & x_3^2 I''_\nu(x_3) \end{vmatrix}, \end{aligned} \tag{4.98}$$

where the numerical prefactor is chosen such that for equal masses the result of Leutwyler is reproduced. The generalization to an arbitrary number of flavors is now obvious. We define a Vandermonde determinant as

$$\Delta(x^2) = \prod_{k<l}(x_k^2 - x_l^2)\ , \tag{4.99}$$

and an $N_f \times N_f$ matrix as

$$A_{kl} = x_k^{l-1} I_\nu^{(l-1)}(x_k) \qquad k,l = 1,\cdots,N_f. \tag{4.100}$$

The partition function is then given as

$$Z_\nu(m_1,\cdots,m_{N_f}) = C_{N_f}\frac{\det A}{\Delta(x^2)}\ , \tag{4.101}$$

where the normalization constant

$$C_{N_f} = 2^{N_f(N_f-1)/2}\prod_{k=1}^{N_f}(k-1)! \tag{4.102}$$

is determined by the limit of equal quark masses.

In section 4, we shall prove this formula for the special case $\nu = 0$. It will be shown that this formula reduces to the Leutwyler-Smilga finite volume partition function for arbitrary ν.

4.5.2 The Itzykson-Zuber integral for complex rectangular matrices

In this section we offer a derivation of the extension of the Itzykson-Zuber integral to the case of arbitrary complex *rectangular* matrices using the diffusion equation method. The result for square matrices has also been obtained in [103]. Our derivation is patterned on the argument for Hermitean matrices, which has been discussed widely in the literature. (See, *e.g.*, [148, 95, 87].) The expression which will be required in section 4 for the calculation of the finite volume partition function for different masses is given in (4.122).

Let σ and ρ be arbitrary complex (*i.e.*, non-Hermitian) $N_1 \times N_2$ matrices. Without loss of generality, we assume that $\nu \equiv N_1 - N_2 \geq 0$. Any such matrix can be diagonalized in the form [119]

$$\sigma = U^\dagger S V \tag{4.103}$$

where

$$S = \begin{pmatrix} \hat{S} \\ \mathbf{0} \end{pmatrix} , \tag{4.104}$$

and where the square diagonal matrix $\hat{S} = \mathrm{diag}(s_1, \cdots, s_{N_2})$ has nonnegative real entries. The matrices (U, V) parameterize the coset space $U(N_1) \times U(N_2)/[U(1)]^{N_2}$. Similarly, ρ can be written as $\rho = U'^\dagger R V'$.

In the following, we evaluate the integral

$$\frac{1}{(\pi t)^{(N_1 N_2)}} \int d\mu(U,V) \, \exp\left(-\frac{1}{t}\mathrm{tr}[(\sigma-\rho)^\dagger(\sigma-\rho)]\right) \tag{4.105}$$

using the diffusion equation method and exploiting the invariance of the measure, $d\mu(U,V)$, which is taken to be the Haar measure of $U(N_1)\times U(N_2)/[U(1)]^{N_2}$. The function

$$F(\rho,t) = \frac{1}{(\pi t)^{(N_1 N_2)}} \int d[\sigma] \exp\left(-\frac{1}{t}\mathrm{tr}[(\sigma-\rho)^\dagger(\sigma-\rho)]\right) \varphi(\sigma) \tag{4.106}$$

with integration measure $d[\sigma] = \prod_{m=1}^{N_1}\prod_{n=1}^{N_2} d\mathrm{Re}\sigma_{mn} d\mathrm{Im}\sigma_{mn}$ satisfies the diffusion equation

$$\sum_{m=1}^{N_1}\sum_{n=1}^{N_2} \frac{\partial^2}{\partial\rho_{mn}\partial\rho^*_{mn}} F(\rho,t) = \frac{\partial}{\partial t} F(\rho,t) \ . \tag{4.107}$$

As initial condition, we choose an invariant function, *i.e.*, $\varphi(\sigma)$ is a function of the eigenvalues of σ only. Then, using the invariance of the measure, we find immediately that $F(\rho, t)$ is a symmetric function of the eigenvalues of ρ.

In order to proceed, we express both the Laplacian and the integration measure in the 'polar coordinates' introduced by the diagonalization (4.103),

$$d[\sigma] = \Omega d[S]\, j^2(S)\, d\mu(U, V) \tag{4.108}$$

where $d[S] = \prod_{n=1}^{N_2} ds_n$. Here, the constant Ω, which depends on the convention adopted for the measure of the group, will be fixed later. The Jacobian is given by $J(S) \equiv j^2(S)$ with

$$j(S) = \prod_{n=1}^{N_2} s_n^{(N_1-N_2)+1/2} \Delta(\hat{S}^2) \ , \tag{4.109}$$

and $\Delta(\hat{S}^2)$ is the Vandermonde determinant defined in (4.99). Because F is an invariant function, it satisfies a diffusion equation which involves only the radial part of the Laplacian:

$$\sum_{n=1}^{N_2} \frac{1}{J(R)} \frac{\partial}{\partial r_n} J(R) \frac{\partial}{\partial r_n} F = 4 \frac{\partial F}{\partial t} \quad . \tag{4.110}$$

This equation can be simplified materially with the introduction of a reduced wave function,

$$f(R, t) = j(R) F(R, t) \quad . \tag{4.111}$$

Now, (4.110) reduces to

$$\sum_{n=1}^{N_2} \left(\frac{\partial^2}{\partial r_n^2} - \frac{1}{j(R)} \left[\frac{\partial^2}{\partial r_n^2} j(R) \right] \right) f(R, t) = 4 \frac{\partial f}{\partial t} \quad . \tag{4.112}$$

Remarkably, this differential equation is separable.[1] Performing the differentiations of $j(R)$, it can be rewritten as

$$\sum_{n=1}^{N_2} \left(\frac{\partial^2}{\partial r_n^2} - \frac{4\nu^2 - 1}{4} \frac{1}{r_n^2} \right) f(R, t) = 4 \frac{\partial f}{\partial t} \ . \tag{4.113}$$

[1] In [103], a separable equation was obtained for $N_1 = N_2$ by the substitution of $\Delta(R^2) F(R, t)$ instead of $j(R) F(R, t)$.

Because of the presence of the factor $j(R)$ in (4.111), $f(R,t)$ is an antisymmetric function of the eigenvalues. Therefore, the solution of (4.113) is given by an integral over a Slater determinant

$$f(R,t) = \int d[S] \, \frac{1}{N_2!} \det_{k,l} |g(r_k, s_l; t)| \; j(S) \, \varphi(S) \; , \tag{4.114}$$

where $g(r,s;t)$ is the kernel of

$$\frac{\partial^2}{\partial r^2} g - \frac{4\nu^2 - 1}{4r^2} g = 4 \frac{\partial g}{\partial t} \quad . \tag{4.115}$$

The kernel can be expressed in terms of the regular eigenfunctions of the Bessel equation of order ν:

$$u'' + [k^2 - (4\nu^2 - 1)/4r^2]u = 0 \quad . \tag{4.116}$$

This is obtained following a separation of variables which leads to a time dependence of the form $\exp(-k^2 t/4)$. The regular eigenfunctions are given by

$$u_k(r) \sim \sqrt{kr} \, J_\nu(kr) \; , \tag{4.117}$$

where J_ν is a Bessel function. Recalling the orthogonality relation for Bessel functions, the kernel of (4.115) can be written as

$$g(r,s;t) = \theta(t) \int_0^\infty dk \, k \sqrt{rs} \mathrm{e}^{-k^2 t/4} \, J_\nu(kr) J_\nu(ks) \quad . \tag{4.118}$$

This can be evaluated to give

$$g(r,s;t) = \theta(t) \, \frac{2}{t} \sqrt{rs} \exp\left(-\frac{r^2 + s^2}{t} \right) I_\nu \left(\frac{2rs}{t} \right) \tag{4.119}$$

where I_ν is a modified modified Bessel function.

Finally, we equate the definition of $F(\rho,t)$ in (4.106) with its expression in terms of the kernel (4.119) as given by (4.111) and (4.114). Since this equality is valid for an arbitrary choice of the initial condition $\varphi(\sigma)$, the integrands of $d[S]$ must be the same.

Hence, we arrive at

$$\int d\mu(U,V) \, \exp\left(-\frac{1}{t} \mathrm{tr}[(\sigma - \rho)^\dagger (\sigma - \rho)] \right)$$

$$= \frac{t^{N_1 N_2 - N_2}}{\prod_{k=1}^{N_2} (r_k s_k)^\nu} \frac{2^{N_2} \pi^{N_1 N_2}}{\Omega} \frac{1}{N_2!} \times \det_{k,l} \left| \exp\left(-\frac{r_k^2 + s_l^2}{t} \right) I_\nu \left(\frac{2 r_k s_l}{t} \right) \right| / \Delta(S^2) \Delta(R^2) \ . \quad (4.120)$$

Here, the value of Ω follows from the normalization integral calculated in [155]:

$$\Omega = \frac{\pi^{N_1 N_2} 2^{N_2}}{\prod_{j=1}^{N_2} j!(j+\nu-1)!} \ , \quad (4.121)$$

where we have used the convention that $\int d\mu(U,V) = 1$.

This result enables us to calculate the Itzykson-Zuber integral for arbitrary complex matrices

$$\begin{aligned} \int d\mu(U,V) \, \exp\left(\mathrm{Re}\, \mathrm{tr}\, U^\dagger S V R \right) &= C_{N_1} C_{N_2} \frac{2^{-\nu(\nu+1)}}{\prod_{k=0}^{\nu-1} k! \prod_{k=1}^{N_2} (r_k s_k)^\nu} \\ &\times \det_{k,l} |I_\nu(r_k s_l)| / \Delta(S^2) \Delta(R^2) \quad (4.122) \end{aligned}$$

When $\nu = 0$, the product of factorials in this expression is understood to be 1. The constants C_{N_1} and C_{N_2} can be evaluated to be

$$C_n = 2^{n(n-1)/2} \prod_{k=1}^{n} (k-1)! \ . \quad (4.123)$$

In the special case $N_1 = N_2$ for which $\nu = 0$, our expression reduces to the result of Guhr and Wettig [103] apart from a normalization constant.

4.5.3 The finite volume partition function for different masses

The finite volume effective partition function of QCD in the sector with topological charge ν, defined in (4.92) and (4.91), can be written as an integral over $U(N_f)$ instead of $SU(N_f)$ [141]

$$Z_\nu = \int_{U(N_f)} d\mu(U) (\det U)^\nu \exp\left(V \Sigma \, \mathrm{Re}\, \mathrm{tr} M U^\dagger \right) \ . \quad (4.124)$$

For $\nu = 0$, this result can be obtained from (4.122) by taking $R = V \Sigma M$ and $S = \mathbf{1}_{N_f}$. Because of the singularity as $S \to \mathbf{1}$, the final result requires a careful analysis of this limit.

In (4.122), we take $S = \mathbf{1} + \delta S$ and expand the modified Bessel functions to order $(\delta S)^{N_f}$. The result is

$$I_\nu(r_k s_l) = \sum_{j=1}^{N_f} \frac{r_k^{j-1}}{(j-1)!} I_\nu^{(j-1)}(r_k)\, \delta s_l^{j-1} \;+\; \mathcal{O}(\delta S^{N_f}) \;, \tag{4.125}$$

which can be written as the product of the matrix A defined in (4.100) and the matrix

$$B_{jl} = \frac{1}{(j-1)!}\, \delta s_l^{j-1} \;. \tag{4.126}$$

The determinant of B can be written as

$$\det(B) = \frac{1}{\prod_{k=1}^{N_f}(k-1)!}\Delta(\delta S) = \frac{1}{C_{N_f}}\Delta\left((1+\delta S)^2\right) \;, \tag{4.127}$$

with the normalization constant defined in (4.123). Hence

$$Z_\nu(M) = C_{N_f}\frac{\det A}{\Delta(R^2)} \;, \tag{4.128}$$

which is simply the result conjectured in (4.101) above. This result is now proved for $\nu = 0$.

We have not proved the result (4.101) for arbitrary ν. However, we offer one non-trivial check of this conjecture by demonstrating that (4.101) reduces to the finite volume partition function of Leutwyler and Smilga in the limit of equal masses. We start with the expression

$$A_{kj} = C_{N_f}^{1/N_f} r_k^{j-1} I_\nu^{(j-1)}(r_k). \tag{4.129}$$

Using the recursion relation

$$\frac{dI_\nu}{dr} = I_{\nu+1} + \frac{\nu}{r} I_\nu \tag{4.130}$$

and adding a suitable multiple of the column to the left of the column in question, we arrive at the matrix

$$A_{kj} \to C_{N_f}^{1/N_f} r_k^{j-1} I_{\nu+j-1}(r_k) \;\; . \tag{4.131}$$

This is evidently correct in going from the first to the second column and, hence, true in general. The fact that the coefficient of $I_{\nu+1}$ in (4.130) is 1 guarantees that the determinant will not be affected by this rearrangement.

To realize the limit $r_k \to r \equiv mV\Sigma$, we write $r_k = r + \delta r_k$ and expand each element in A in a Taylor series through order $\delta r_k^{N_f-1}$. The result is that

$$A_{kj} = C_{N_f}^{1/N_f} \sum_{p=1}^{N_f} \frac{1}{(p-1)!} \frac{d^{p-1}}{dr^{p-1}} \left[r^{j-1} I_{\nu+j-1}(r) \right] \, \delta r_k^{p-1} \ . \tag{4.132}$$

This can be written as the product of the matrix with elements

$$M_{jp} = C_{N_f}^{1/N_f} \frac{d^{p-1}}{dr^{p-1}} \left[r^{j-1} \, I_{\nu+j-1}(r) \right] \tag{4.133}$$

and the matrix $B_{jp} \equiv (\delta r_k)^{p-1}/(p-1)!$. As in (4.127), we have

$$\det(B) = \prod_{p=1}^{N_f} \frac{1}{(p-1)!} \Delta(\delta R) = \frac{1}{C_{N_f}} \left(\frac{1}{r} \right)^{N_f(N_f-1)/2} \quad \Delta((R+\delta R)^2) \ . \tag{4.134}$$

The determinant of the matrix M can be simplified by using the recursion relation

$$\frac{dI_\nu}{dr} = I_{\nu-1} - \frac{\nu}{r} I_\nu \tag{4.135}$$

and adding a suitable multiple of the row immediately above the row in question. As before, this rearrangement does not alter the determinant. This leaves us with

$$M_{kj} \to C_{N_f}^{1/N_f} r^{j-1} I_{\nu+j-k} \ . \tag{4.136}$$

Now, the factor r^j can be extracted from each column and used to eliminate the r dependence in the prefactor. Thus, we arrive at the final result

$$\det(A) = \det(M) \det(B) = \det_{k,j} I_{\nu+j-k} \ , \tag{4.137}$$

which is precisely the result in [141] as given in (4.93).

4.5.4 Conclusions

We have obtained the finite volume QCD partition function for different quark masses in the range $1/\Lambda \ll L \ll 1/m_\pi$. This result generalizes the finite volume partition function obtained previously by Leutwyler and Smilga for the case of equal quark masses. In order to derive this result, we were led to generalize the Itzykson-Zuber integral to arbitrary rectangular complex

matrices. The integral for square matrices, first obtained by Guhr and Wettig, leads immediately to the proof in the sector of zero topological charge. Based on the result for two flavors and the general result for $\nu = 0$, we have conjectured the result for arbitrary ν and N_f. As a decidedly nontrivial check of this conjecture, we have shown that the result of Leutwyler and Smilga emerges in the limit of equal quark masses.

We wish to note a remarkable coincidence. Consider the Itzykson-Zuber formula for complex rectangular matrices (with ν equal to the difference between the number of rows and columns) in the same limit considered for $\nu = 0$. Up to a factor, this leads to the finite-volume partition function in the sector of topological charge ν. While we can offer no explanation of this coincidence, it may be useful to note that, in the chiral limit, the joint eigenvalue density of the random matrix model associated with the finite volume partition function depends only on the combination $\nu + N_f$ [204, 201, 202].

4.6 Universality of higher spectral correlation functions for non-invariant deformations of gaussian chiral random matrix models

As explained in the beginning of the previous section this work is both the motivation for and the natural continuation of the work presented in the previous section. The chiral Itzykson-Zuber integrals are used to analytically compute the higher correlation functions of the temperature deformed (and therefore non-invariant) chiral gaussian unitary ensemble.

The main result of this section is the universality of the aforementioned correlation functions except for a trivial scaling of their functional form given by a temperature dependent factor. This result is valid up to a critical temperature as discussed in section 4.3.

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We address the question of universality within the context of random matrix theory. An observable, typically a spectral correlation function, will be called universal if it is stable against deformations of the probability distribution. Recently, this topic has attracted a great deal of attention. Two types of deformations have been considered: deformations which maintain the invariance of the random matrix ensembles [106, 11, 118, 63, 128] and those which violate that invariance [216, 12, 44, 45, 41, 25, 98, 99, 230]. We will consider spectral correlation functions for an example of the latter class. In

all cases which have been studied, spectral correlations measured in units of the average level spacing are found to be universal even for deformations that change the spectral density on a macroscopic scale.

Specifically, we investigate the stability of the spectral correlations of the chiral ensembles [180, 203, 204, 201, 202]. They are constructed to describe the fluctuations of the Dirac eigenvalues in lattice QCD [113, 110] and are relevant to the theory of universal conductance fluctuations. Because of an underlying $U_A(1)$ symmetry (i.e. because the Dirac operator commutes with γ_5) , the eigenvalues of chiral matrices occur in pairs $\pm\lambda$. Therefore, one can consider three types of universal behavior each of which is given by invariant random matrix ensembles: (i) correlations in the bulk of the spectrum [86, 155, 196], (ii) correlations near the edge of the spectrum [135], and (iii) correlations near $\lambda = 0$. In this case, the microscopic spectral density (i.e., the spectral density near zero on the scale of a typical eigenvalue spacing) is universal. This has been illustrated for both invariant [42, 159, 3] and non-invariant deformations [125].

The invariant random matrix ensembles and the chiral random matrix ensembles are part of a larger classification scheme: Altland and Zirnbauer have shown that there is a one to one correspondence between random matrix ensembles and symmetric spaces [67, 8].

The microscopic spectral density is of immediate physical interest. According to the Banks-Casher formula [22], the spectral density at zero is directly proportional to the order parameter of the chiral phase transition in QCD (i.e., the chiral condensate). The microscopic spectral density provides information regarding the approach to the thermodynamic limit. This has been demonstrated, for example, in connection with the dependence of the chiral condensate on the valence quark mass [206]. Recently, it has been verified by direct lattice calculations that the microscopic spectral density of lattice QCD is given by one of the chiral ensembles [220]. Moreover, the microscopic spectral density enter in sum-rules [141] for the inverse eigenvalues of the Dirac operator.

The model which we shall consider is the chiral unitary ensemble perturbed by the lowest Matsubara frequencies, $\pm\pi T$. This model was introduced in [127] as a model for the chiral phase transition. Indeed, the average spectral density for this model undergoes a transition from one semicircle at zero temperature to two disjunct semicircles at high temperature. In [125] it was shown that the spectral density of this model on the scale of individual average level spacings is independent of the temperature.

The aim of the present paper is to show that, below to the critical temperature, *all* spectral correlations measured in units of the average eigenvalue

spacing are independent of the temperature. This will be achieved using the methods of super-symmetric integration introduced by Guhr [95] and certain super-symmetric Itzykson-Zuber integrals [122, 148, 103, 124]. At zero temperature our results coincide with earlier work [180, 203, 204, 201, 202, 181, 153, 154, 217, 159, 3].

The outline of the paper is as follows. In section 4.6.1, we introduce the random matrix model and express the correlation functions in terms of a partition function. In section 4.6.2, we reduce this partition function to an integral over a supermatrix, σ, of much smaller size. The choice of an explicit parametrization for σ is essential for the rest of the paper. Thus, we will argue in section 4.6.3 that the correlation functions can be obtained by deforming the non-compact parametrization required for uniform convergence into a compact parametrization of σ. This will be done through a detailed investigation of the one-point function. In section 4.6.4, we show that all correlation functions follow from a two-point kernel. This will be evaluated at zero temperature in section 4.6.5, and we will reproduce the well-known Laguerre kernel. In section 4.6.6, we evaluate the two-point kernel at finite temperature using a saddle point approximation. Our primary result will be that the correlation functions are independent of temperature except for a trivial rescaling of their arguments.

4.6.1 The random matrix model and the partition function for correlations

The partition function of Euclidean QCD is given by

$$Z_E = \langle \det(i\gamma \cdot D + iM) \rangle_{S_E} \tag{4.138}$$

where $\gamma \cdot D$ is the Dirac operator, M is the mass matrix, and the brackets denote the average over gauge field configurations with respect to the Euclidean Yang-Mills action S_E. Using a lattice regularization with anti-periodic boundary conditions in the time direction, we can expand the fermion fields as a sum over Matsubara frequencies to obtain

$$\psi(\mathbf{x}, \tau) = \sum_{k=1}^{N} \sum_{l=-n+1}^{n} \phi_k(\mathbf{x}) \exp\left(\pi i (2l-1) T \tau\right) \ . \tag{4.139}$$

Here, T is the temperature (i.e., the inverse length of the time axis), N is to be identified with the volume of space, and n denotes the total number of Matsubara frequencies retained in the expansion. The ϕ_k are properly normalized spinors.

Using this basis, we separate the time derivative in the Dirac operator, $\gamma_0\partial_\tau$, from the remaining terms. In terms of the matrix elements of the Dirac operator the partition function for N_f massless flavors can then be written as

$$Z_{\rm gauge} = \left\langle \det{}^{N_f} \left[\begin{pmatrix} 0 & iC^\dagger \\ iC & 0 \end{pmatrix} + \begin{pmatrix} 0 & i\Theta\otimes \mathbf{1}_N \\ i\Theta\otimes \mathbf{1}_N & 0 \end{pmatrix} \right] \right\rangle_{S_E} \quad . \tag{4.140}$$

The identity matrix of size N is denoted by $\mathbf{1}_N$, and Θ contains the Matsubara frequencies $\Theta = {\rm diag}\,(-(2n-1)\pi T, \ldots, -\pi T, \pi T, \ldots, (2n-1)\pi T)$. In (4.140), we have used a chiral representation of the Dirac matrices. For fundamental fermions with three colors, the matrix C is a complex matrix of size $2nN \times 2nN$. The detailed form of C depends on the particular gauge field configuration.

The corresponding random matrix model is obtained by replacing the matrix elements of C by independent random variables with a Gaussian distribution. Thus, instead of (4.140), we study the properties of

$$Z_{\rm RM} = N_D \int d[C] e^{-N\Sigma^2 {\rm Tr}\, C^\dagger C} \det{}^{N_f} \left[\begin{pmatrix} 0 & iC^\dagger \\ iC & 0 \end{pmatrix} + \begin{pmatrix} 0 & i\Theta\otimes \mathbf{1}_N \\ i\Theta\otimes \mathbf{1}_N & 0 \end{pmatrix} \right]$$

where the measure $d[C]$ is the Haar measure defined by $\prod_{m,n} d({\rm Re}\, C_{mn}) d({\rm Im}\, C_{mn})$, and N_D is a normalization constant to be specified later. We wish to stress that this partition function is a *schematic* model of the QCD partition function. For example, we have ignored all spatial dependence of the matrix elements of the Dirac operator, and the critical exponents of this model are necessarily those of mean field theory. It is our claim, however, that this partition function belongs to the same universality class as the QCD partition function with respect to local spectral fluctuations.

In this paper, we will study the spectral correlation functions of the model given by (4.141). For simplicity, we will restrict our attention to the case in which we retain only the lowest Matsubara frequencies, $\pm\pi T$. This enables us to replace Θ by πT using a unitary transformation. After redefining N, the matrix C can be taken to be a complex $N \times N$ matrix.

As was shown in [127, 125], this model shows a second-order phase transition at $\pi T\Sigma = 1$. Below this temperature, chiral symmetry is broken spontaneously with the chiral condensate given by $\Xi = \Sigma\sqrt{(1-\pi^2 T^2\Sigma^2)}$. At all temperatures, the spectral density follows from the solution of a cubic equation [127, 185, 125]. In particular, we mention that, according to the Banks-Casher formula [22], the spectral density at zero virtuality is simply related to the chiral condensate:

$$\Xi = \frac{\pi\rho(0)}{N} \quad . \tag{4.141}$$

In [125] it was shown that the microscopic spectral density, defined as

$$\rho_S(u) = \lim_{N\to\infty} \frac{1}{N\Xi}\rho(\frac{u}{N\Xi}) \tag{4.142}$$

is independent of the temperature parameter in the random matrix model. This result greatly adds to our understanding of the empirically observed universality of the microscopic spectral density in lattice QCD [206, 220] and instanton liquid simulations [180, 203, 204, 201, 202]. Because of these results and numerical results for higher-order correlation functions in the bulk of lattice QCD Dirac spectra [113, 110], we expect that the higher-order microscopic correlation functions are universal as well.

The spectral density of a matrix with eigenvalues λ_k is given by

$$\rho(x) = \sum_k \delta(x - \lambda_k) \ . \tag{4.143}$$

The k-level correlation functions are then defined as

$$R_k(x_1, \ldots, x_k) = \langle \rho(x_1) \ldots \rho(x_k) \rangle \ , \tag{4.144}$$

where $\langle \cdots \rangle$ denotes the ensemble average with respect to (4.141). However, in our approach is is more convenient to work with correlation functions of the resolvent

$$G(x) = \left\langle \mathrm{Tr}\, \frac{-1}{x + i\epsilon - D} \right\rangle \ , \tag{4.145}$$

where D is the random matrix Dirac operator

$$D = \begin{pmatrix} 0 & C + \pi T \\ C^\dagger + \pi T & 0 \end{pmatrix}, \tag{4.146}$$

and ϵ is a positive infinitesimal. The spectral density is then given by

$$\rho(x) = \frac{1}{\pi}\mathrm{Im}\, G(x). \tag{4.147}$$

The corresponding correlation functions are defined by

$$\hat{R}_k(x_1, \ldots, x_k) = \left\langle \prod_{l=1}^{k} \frac{1}{\pi}\mathrm{Tr}\, \frac{-1}{x_l + i\epsilon_l - D} \right\rangle \ . \tag{4.148}$$

The spectral correlation functions follow by taking the imaginary part of each trace. In the following, we will drop the $i\epsilon_l$ term, assuming that x_l has a positive infinitesimal imaginary part.

Using the identity

$$\mathrm{Tr}\, A^{-1} = -\frac{1}{2}\frac{\partial}{\partial j}\left[\det(A-j)/\det(A+j)\right]\big|_{j=0} \ , \tag{4.149}$$

one can express the correlation functions in terms of a generating function

$$\hat{R}_k(x_1,\ldots,x_k) = \frac{1}{(2\pi)^k}\prod_{l=1}^{k}\frac{\partial}{\partial j_l} Z_k(j_1,\ldots,j_k)\Big|_{j_l=0} \ , \tag{4.150}$$

$$Z_k(j_1,\ldots,j_k) = N_D \int d[C] e^{-N\Sigma^2 \mathrm{Tr}\, C^\dagger C} \det{}^{N_f} D \prod_{l=1}^{k}\frac{\det(D-x_l+j_l)}{\det(D-x_l-j_l)} \ . \tag{4.151}$$

The prefactor, N_D, in (4.151) is to be chosen such that $Z_k(0) = 1$. In the rest of the paper, we will restrict our attention to the quenched problem with $N_f = 0$. This is due to the fact that the current method does not allow us to find a solution for arbitrary N_f. The reason for this will be discussed at the end of section 5.

4.6.2 Reduction of the partition function

The expression (4.151) for the generating function of correlators involves an integral over a matrix, C, with $2N^2$ degrees of freedom. When the deterministic part, Θ, is absent, we can exploit the unitary invariance

$$C \to UCV^{-1}, \tag{4.152}$$

with U and V unitary matrices, in order to rewrite the partition function as an integral over the eigenvalues of C only. It can then be evaluated easily using, for example, the orthogonal polynomial method [149, 181, 153, 154]. The presence of Θ destroys unitary invariance, and standard methods are unsuitable. However, unitary invariance is not essential in the supersymmetric formulation of random matrix theory [76, 216]. Below, we adapt an approach based on the supersymmetric method which is described in [95]. This method exploits the determinantal structure of the correlation functions and allows us to calculate all k-point correlation functions at the same time. Readers not familiar with the use of supersymmetry methods in random matrix theory are referred to [76, 216, 95].

In this section, we will express the partition function, (4.151), as an integral over a $4k \times 4k$ supermatrix in which N appears only as an overall factor

in the action. This is particularly useful for the investigation of the thermodynamic limit (i.e., $N \to \infty$).

The ensemble average in (4.151) is performed by writing the determinants as Gaussian integrals over commuting and anticommuting variables [216]. Equivalently, the product of determinants and inverse determinants in (4.151) can be considered as a superdeterminant and can be written as a Gaussian integral over a complex supervector. In order to do this we introduce the following supermatrices

$$\begin{aligned} \mathbf{C} &= \mathrm{diag}(C,\ldots,C;C,\ldots,C)\ , \\ \mathbf{x} &= \mathrm{diag}(x_1,\ldots,x_k;x_1,\ldots,x_k)\ , \\ \mathbf{j} &= \mathrm{diag}(-j_1,\ldots,-j_k;j_1,\ldots,j_k)\ . \end{aligned} \tag{4.153}$$

Here, a semicolon separates the k boson-boson blocks from the k fermion-fermion blocks. Each block is of size $N \times N$. Further, x_p and j_p are understood to be multiplied by the $N \times N$ identity matrix.

In order to exploit the chiral structure of the problem, we introduce a pair of complex supervectors, ϕ_A and ϕ_B, each of size $(k\,N; k\,N)$. The two numbers refer to the number of commuting and anticommuting components. Then, the product of determinants in (4.151) can be written as

$$\begin{aligned} \prod_{l=1}^{k} \frac{\det(D - x_l + j_l)}{\det(D - x_l - j_l)} &= \mathrm{sdet}^{-1} \left| \begin{array}{cc} -\mathbf{x}+\mathbf{j} & \mathbf{C}^\dagger + \pi T \\ \mathbf{C} + \pi T & -\mathbf{x}+\mathbf{j} \end{array} \right| \\ &= \int d[\phi_A] d[\phi_B] \exp\left[-i \begin{pmatrix} \phi_A^\dagger \\ \phi_B^\dagger \end{pmatrix} \right. \\ &\times \left. \begin{pmatrix} -\mathbf{x}+\mathbf{j} & \mathbf{C}^\dagger + \pi T \\ \mathbf{C} + \pi T & -\mathbf{x}+\mathbf{j} \end{pmatrix} \begin{pmatrix} \phi_A \\ \phi_B \end{pmatrix} \right] , \end{aligned} \tag{4.154}$$

where we have taken $\Theta = \pi T$. In (4.154), the blocks inside the superdeterminant refer to the chiral structure, i.e., each block is itself a supermatrix. The measure on the right hand side of (4.154) is

$$d[\phi] = \prod_{p=1}^{k} \left[\prod_{l=1}^{N} \frac{d(\phi_{pl}^0) d(\phi_{pl}^0)^*}{2\pi} \right] \prod_{q=1}^{k} \left[\prod_{m=1}^{N} \frac{d(\phi_{qm}^1) d(\phi_{qm}^1)^*}{i} \right] . \tag{4.155}$$

Here, ϕ refers to either ϕ_A or ϕ_B; ϕ_{pl}^0 and ϕ_{qm}^1 denote the commuting and anticommuting components of ϕ, respectively. The ranges of the indices follow from the limits in the products. The integrals with respect to the anticommuting variables are normalized according to the convention [32], $\int d\chi\, \chi = 1$.

The constants in (4.155) are chosen so that Gaussian integrals do not result in additional prefactors. We recall that x_l is understood to have a positive imaginary part, which is sufficient to ensure the convergence of (4.154).

The ensemble average can now be performed immediately by substituting (4.154) into (4.151) and completing the square in the exponent. Interchange of the orders of the C- and ϕ-integration is justified because the C-integral is uniformly convergent in ϕ. In order to proceed, we rewrite the terms appearing in the exponent of (4.154):

$$\phi_A^\dagger \cdot \mathbf{C}^\dagger \cdot \phi_B = \sum_{\epsilon=0}^{1} \sum_{p=1}^{k} (-1)^\epsilon \mathrm{Tr}\ C^\dagger \cdot \left(\phi_{Bp}^\epsilon \otimes \phi_{Ap}^{\epsilon\dagger} \right) \ , \tag{4.156}$$

and the complex conjugate equation for $\phi_B^\dagger \cdot \mathbf{C} \cdot \phi_A$. Here, ϕ_p^0 and ϕ_q^1 are commuting and anticommuting vectors of length N which represent the components of ϕ_A and ϕ_B. The C-integral can now be performed using

$$\begin{aligned} \int d[C] \exp\left[-N\Sigma^2 \mathrm{Tr}\, C^\dagger C - i\, \mathrm{Tr}\, (C^\dagger X + CY) \right] \\ = N_C^{-1} \exp\left[-\frac{1}{N\Sigma^2} \mathrm{Tr}\, (XY) \right] , \end{aligned} \tag{4.157}$$

where $N_C^{-1} = \int d[C] \exp\left[-N\Sigma^2 \mathrm{Tr}\, C^\dagger C \right]$. (For $N_f = 0$ we have $N_C = N_D$ in (4.151).) The matrices are arbitrary complex $N \times N$ matrices. Thus, the generating function (4.151) becomes

$$Z_k(j_1, \ldots, j_k) =$$

$$\begin{aligned} \int d[\phi_A] d[\phi_B]\ \exp\left[-\frac{1}{N\Sigma^2} \sum_{\epsilon\epsilon'=0}^{1} (-1)^{\epsilon'} \sum_{pp'=1}^{k} (\phi_{Ap}^{\epsilon\dagger} \cdot \phi_{Ap'}^{\epsilon'}) (\phi_{Bp'}^{\epsilon'\dagger} \cdot \phi_{Bp}^{\epsilon}) \right] \\ \times \exp\left[i \sum_{\epsilon=0}^{1} (-1)^\epsilon \Big(\sum_{p=1}^{k} (\mathbf{x} - \mathbf{j})_p (\phi_{Ap}^{\epsilon\dagger} \phi_{Ap}^{\epsilon} + \phi_{Bp}^{\epsilon\dagger} \phi_{Bp}^{\epsilon}) - \pi T (\phi_{Ap}^{\epsilon\dagger} \phi_{Bp}^{\epsilon} + \phi_{Bp}^{\epsilon\dagger} \phi_{Ap}^{\epsilon}) \Big) \right] . \end{aligned} \tag{4.158}$$

As a result of the ensemble average, we obtain a fourth-order term in the exponent which should be decoupled using a Hubbard-Stratonovitch transformation. In order to accomplish this, we rewrite the first exponential in (4.158) as $\exp\left(-\mathrm{str}\,(AB)/N\Sigma^2\right)$. The $(k+k) \times (k+k)$ super-matrix A is given by

$$A_{lm}^{\epsilon\epsilon'} = (\phi_{Al}^{\epsilon\dagger} \cdot \phi_{Am}^{\epsilon'}) \tag{4.159}$$

with a similar form for B. Now, we can 'undo' the transformation in (4.157) by introducing a non-hermitean complex $(k+k)\times(k+k)$ super-matrix, σ.

$$\exp\left(-\frac{1}{N\Sigma^2}\mathrm{str}\,(AB)\right) = \int d[\sigma]\,\exp\Big[-N\Sigma^2\mathrm{str}\,\left(\sigma^\dagger\cdot\sigma\right) -i\,\mathrm{str}\,\left(\sigma^\dagger\cdot A+\sigma\cdot B\right)\Big]. \qquad (4.160)$$

After writing $\mathrm{str}\,(\sigma^\dagger\cdot A)=\phi_A^\dagger\cdot(\sigma^\dagger\otimes\mathbf{1}_N)\cdot\phi_A$ and $\mathrm{str}\,(\sigma\cdot A)=\phi_B^\dagger\cdot(\sigma\otimes\mathbf{1}_N)\cdot\phi_B$, and performing a shift of integration variables given by $\sigma\to\sigma+\mathbf{x}-\mathbf{j}$, we can express the generating function for the correlation functions as

$$\begin{aligned} Z_k(j_1,\ldots,j_k) &= \int d[\phi_A]d[\phi_B]\int d[\sigma] \\ &\quad\times\exp\left\{-N\Sigma^2\mathrm{str}\,\left[(\sigma^\dagger+\mathbf{x}-\mathbf{j})\cdot(\sigma+\mathbf{x}-\mathbf{j})\right]\right\} \\ &\quad\times\exp\left\{-i\phi_A^\dagger\cdot(\sigma^\dagger\otimes\mathbf{1}_N)\cdot\phi_A-i\phi_B^\dagger\cdot(\sigma\otimes\mathbf{1}_N)\cdot\phi_B\right. \\ &\quad\left.-i\pi T(\phi_A^\dagger\cdot\phi_B+\phi_B^\dagger\cdot\phi_A)\right\}. \qquad (4.161) \end{aligned}$$

Here we still use the definition (4.153) for $\mathbf{x}$ and $\mathbf{j}$, however from this point on x_p and j_p in (4.153) are *not* meant to be multiplied by the $N\times N$ identity matrix.

At this point, we would like to change the order of the integrations in (4.161) and perform the Gaussian integrals in ϕ. This would result in a superdeterminant involving the matrix σ. However, this is possible only if the ϕ-integral is uniformly convergent in σ, which is not the case if σ is an arbitrary complex matrix. However, it can be shown in general [232] that it is possible to choose a certain non-compact parametrization of the σ-variables that ensures uniform convergence. An explicit construction for one- and two-point correlation functions has been given in [17, 125]. The parametrization in the case of the one-point function will be discussed in great detail in the next section.

Postponing further discussion of these points, we arrive at the following expression after a change in the order of the integrals in (4.161)

$$\begin{aligned} Z_k(\mathbf{j}) &= \int d[\sigma]\exp\left(-N\Sigma^2\mathrm{str}\,(\sigma^\dagger+\mathbf{x}-\mathbf{j})\cdot(\sigma+\mathbf{x}-\mathbf{j})\right) \\ &\quad\times\mathrm{sdet}^{-N}\begin{vmatrix}\sigma^\dagger & \pi T\\ \pi T & \sigma\end{vmatrix}. \qquad (4.162) \end{aligned}$$

This partition function, with N appearing only as an overall parameter, is amenable to a saddle point approximation.

Before we continue with the evaluation of this partition function, we will show in the case of the one-point function that the imaginary part of the partition function can be obtained by replacing the non-compact parametrization of (4.162) by a compact parametrization. We will conjecture that all higher-order multipoint correlators R_k, as opposed to the $\hat{R}_k$ defined in (4.148), can also be obtained by using a compact parametrization with the σ-variables parameterized according to

$$\sigma = USV^{-1} \ . \tag{4.163}$$

This choice of integration domain allows us to utilize the Itzykson-Zuber-like integrals developed earlier [122, 148, 103, 124] for the integration over the super-unitary matrices. Unfortunately, we have not been able to construct a rigorous proof of this statement.

4.6.3 The one-point function

In this section, we wish to make the point that the spectral density can be obtained from a compact parametrization of the σ-matrix rather than the non-compact parametrization which is required for uniform convergence, and which justifies the interchange of ϕ- and σ-integrations. Since fermionic integrals are always finite, convergence problems arise only from the integrations over the boson-boson block of the σ matrix.

The uniform convergence of the ϕ-integrations can be achieved if we perform a Hubbard-Stratonovitch transformation using the identity

$$e^{-a^2+b^2} = \frac{-1}{\pi i} \int_{-\infty}^{\infty} ds \int_{-\infty}^{\infty} \sigma d\sigma e^{-\sigma^2 - 2ia\sigma \cosh s - 2ib\sigma \sinh s} \ , \tag{4.164}$$

where a is real positive and $b^2 - a^2$ has a negative real part. When expressed in terms of the real and imaginary parts of $\sigma_{BB} = \sigma_1 + i\sigma_2$, this identity corresponds to the non-compact parametrization introduced in [125]. This parametrization is given as

$$\begin{aligned} \sigma_1 &= (\sigma - i\epsilon) \cosh s/\Sigma \ , \\ \sigma_2 &= i(\sigma - i\epsilon) \sinh s/\Sigma \ , \end{aligned} \tag{4.165}$$

where both σ and s run over the real line. Note that this parametrization covers only half of the (σ_1, σ_2) plane. (The parameter σ introduced here should not be confused with the matrix σ defined in the previous section.) We wish to

contrast this parametrization with the compact parametrization (i.e., in polar coordinates),

$$\begin{aligned} \sigma_1 &= \sigma \cos\theta/\Sigma \ , \\ \sigma_2 &= \sigma \sin\theta/\Sigma \ . \end{aligned} \tag{4.166}$$

After performing the integration over the Grassmann variables in the σ-matrix, the partition function can be written as

$$\begin{aligned} Z(j) &= \frac{N^2}{\pi^2} \int_{-\infty}^{\infty} \sigma d\sigma \int_{-\infty}^{\infty} ds \int_0^{\infty} \rho d\rho \int_0^{2\pi} d\varphi F((\sigma - i\epsilon)^2, \rho^2, t^2) \\ &\times \left(\frac{\rho^2 + t^2}{t^2 - (\sigma - i\epsilon)^2} \right)^N \\ &\times e^{-N[\sigma^2 + \rho^2 + 2(x-j)\Sigma(\sigma - i\epsilon)\cosh s + 2ix\Sigma\rho\cos\varphi + \Sigma^2((x-j)^2 - x^2)]} \ , \end{aligned} \tag{4.167}$$

where $t = \pi T \Sigma$, ρ, and φ parameterize the fermion-fermion block of the σ matrix. We have displayed the $i\epsilon$ dependence of (4.167) explicitly. The convergence of the s-integral is guaranteed by choosing $x - j$ to be pure imaginary. The definition of the function $F((\sigma - i\epsilon)^2, \rho^2, t^2)$ and additional details can be found in [125].

We perform the σ-integral first by saddle-point integration. The σ-integrand has poles at $\pm t + i\epsilon$, which allows us to deform the integration contour into the lower complex half-plane. The saddle-points in the σ integration are given by $\bar{\sigma} = \pm i\sqrt{1 - t^2}$, and thus only the saddle point with the negative sign can be reached by deforming the integration contour. The resulting integral can be analytically continued to $x - j$ just above the real axis.

We now consider the imaginary part of the partition function. The φ integral gives rise to a Bessel function, J_0. Thus, the only contribution to the imaginary part comes from the factor

$$S = \int_{-\infty}^{\infty} ds \exp[-2N(x - j)\Sigma\bar{\sigma}\cosh s] \ . \tag{4.168}$$

For positive x, we change integration variables according to $s \to s - \pi i/2$. Then, the integration path can be deformed into the integration path I shown in Fig. 1, and S is given by

$$S = \int_{\mathrm{I}} ds \exp[2iN(x - j)\Sigma\bar{\sigma}\sinh s] \ . \tag{4.169}$$

The imaginary part of S is given by half the sum (note the arrows) of the contributions along the integration paths I and II in Fig. 1. The integral along

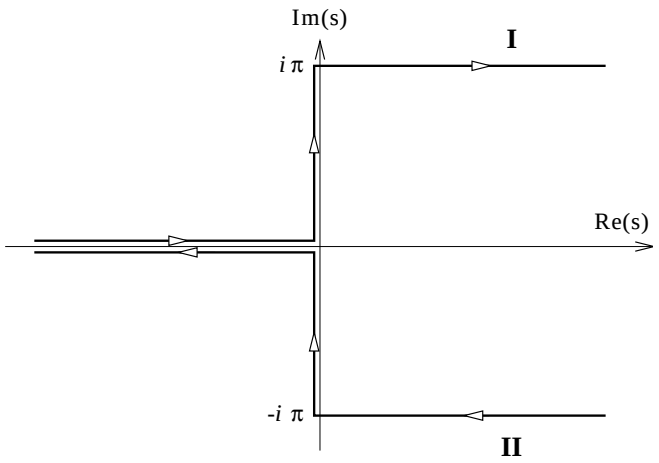

Figure 4.1: The integration contour for the imaginary part of the generating function.

the parts of the integration contour parallel to the real axis cancel, and we are left with

$$\mathrm{Im}\, S = \frac{1}{2}(S - S^*) = \mathrm{Im}\, \frac{1}{2}\int_{-\pi i}^{\pi i} ds \exp[2iN(x-j)\Sigma\bar{\sigma}\sinh s] \ . \qquad (4.170)$$

If we change integration variables, $s = i\theta$, we find

$$\mathrm{Im}\, S = \int_0^{\pi} d\theta \cos[2iN(x-j)\Sigma\bar{\sigma}\sin\theta] \ . \qquad (4.171)$$

In effect, this implies that the imaginary part of the partition function can be obtained by replacing the (σ, s) parametrization required for uniform convergence by a compact parametrization such as (4.166). With this parametrization, the partition function is given as

$$\begin{aligned} Z^C(j) &= \frac{N^2}{\pi^2}\int_0^{\infty}\sigma d\sigma\int_0^{2\pi}d\theta\int_0^{\infty}\rho d\rho\int_0^{2\pi}d\varphi F((\sigma-i\epsilon)^2,\rho^2,t^2) \\ &\times \left(\frac{\rho^2+t^2}{t^2-(\sigma-i\epsilon)^2}\right)^N \\ &\times e^{-N(\sigma^2+\rho^2+2(x-j)\Sigma(\sigma-i\epsilon)\cos\theta+2ix\Sigma\rho\cos\varphi+\Sigma^2((x-j)^2-x^2))}. \end{aligned} \qquad (4.172)$$

In this form, the imaginary part of the partition function is determined by the $i\epsilon$ in the singularities of the pre-exponential factor. Therefore, the imaginary part is even in σ. This allows us to extend the σ-integration from $-\infty$ to ∞ with the introduction of a factor $1/2$. Such an extension of the σ-integration

to the full real axis is required if we are to perform this integral by saddle-point integration. This integration can be carried out in the same fashion as in the non-compact case and yields the same result for $\mathrm{Im}\, Z^C(j)$ as obtained by substituting the final result for $\mathrm{Im}\, S$ (4.171) back into $\mathrm{Im}\, Z(j)$ defined in (4.167).

By contrast, the real part of the integral is odd in σ, and it is not possible to extend the σ-integration over the entire real axis. Consequently, the σ-integration for the real part of $Z(j)$ cannot be performed by saddle-point integration if a compact parametrization is adopted; only the imaginary part of $Z(j)$ can be obtained in this fashion. A similar phenomenon was observed in [17] for both the one- and two-point functions.

This leads us to the following conjecture: The spectral correlation functions, which are given by the products of the imaginary parts of the resolvents at different points, can be obtained by parameterizing the σ-integration in (4.162) as

$$\sigma = U(S - i\epsilon)V^{-1} \ , \tag{4.173}$$

where U and V are taken from the super-unitary group as discussed in the next section and where S is a diagonal matrix. This conjecture is central to the remainder of this paper.

4.6.4 Integration over soft modes using Itzykson-Zuber integrals

We now return to the evaluation of the generating function for correlations. One possible method for the evaluation of (4.162) would be to integrate out anticommuting degrees of freedom first. This is simple for a small number of Grassmann variables when only a small number of terms contribute to the integral. However, even for k as small as 2 this approach is impractical [17], and for larger values of k it is virtually impossible to perform the integrals this way.

Rather, we adopt a technique developed in [95], where it was applied to a study of the Gaussian unitary ensemble. This technique consists essentially of separating the eigenvalue and angular coordinates of σ by diagonalization and then integrating over the angular coordinates using a super-symmetric analogue of the Itzykson-Zuber integral. The remaining bosonic eigenvalue integrals can then be carried out either exactly (e.g., for the case of zero temperature considered in section 5) or in saddle point approximation (e.g., for non-zero temperature as in section 6.) The main virtue of Guhr's technique

is that it preserves the determinantal structure of the correlation functions, which makes it possible to obtain all correlation functions at the same time.

We begin by reminding the reader of some relevant properties of super-unitary matrices. As in the case of an arbitrary complex matrix [119], an arbitrary super-matrix can be diagonalized by two super-unitary matrices

$$\sigma = U^\dagger S V \ , \tag{4.174}$$

where $S = \mathrm{diag}(s_1^0, \ldots, s_k^0; is_1^1, \ldots, is_k^1)$ is a diagonal matrix which can be taken to have non-negative real entries and (U, V) parameterize the group $(U(k|k) \times U(k|k)) / [U(1)]^{k+k}$. (Here, $U(k|k)$ denotes the super-unitary group) With this parametrization, the integration measure can be written as [2]

$$d[\sigma] = d[S^2] d\mu(U, V) B^2(S^2) \ , \tag{4.175}$$

where $d\mu(U, V)$ is the super-invariant Haar measure, $d[S^2] = \prod_{\epsilon=0}^{1} \prod_{l=1}^{k} d(s_l^\epsilon)^2$. The Jacobian of the transformation is given by the square of the Berezinian [103]. When the bosonic and fermionic blocks are of the same size (i.e. for $N_f = 0$) the Berezinian can be simplified to [95]

$$B(S^2) = \det \left[\frac{1}{(s_p^0)^2 - (is_m^1)^2} \right]_{p,m=1,\ldots,k} . \tag{4.176}$$

In terms of integration variables defined by the polar decomposition (4.174) (with an infinitesimal negative imaginary increment included in the eigenvalues), the generating function (4.162) can be rewritten as

$$\begin{aligned} Z_k(\mathbf{j}) &= \int d[S^2] \, B^2(S^2) \, \mathrm{sdet}^{-N} \begin{vmatrix} S & \pi T \\ \pi T & S \end{vmatrix} \\ &\times \int d\mu(U, V) \exp\left(-N \Sigma^2 \mathrm{str} \, (\sigma + \mathbf{x} - \mathbf{j})^\dagger (\sigma + \mathbf{x} - \mathbf{j}) \right) , \end{aligned} \tag{4.177}$$

where we have used the fact that

$$\mathrm{sdet} \begin{vmatrix} \sigma^\dagger & \pi T \\ \pi T & \sigma \end{vmatrix} = \mathrm{sdet} \begin{vmatrix} S & \pi T \\ \pi T & S \end{vmatrix} . \tag{4.178}$$

The group integral in (4.177) has precisely the form of an Itzykson-Zuber integral over complex super-unitary matrices. For arbitrary complex super-matrices $\sigma = U^{-1} S V$ and $\rho = U'^{-1} R V'$, one finds by an application of the

[2] Note that our normalization of $d[\sigma]$ and $d\mu(U, V)$ in (4.179) differs by a factor 2^{2k^2} from ref. [103].

heat kernel method [122, 148, 103, 124] that

$$\int d\mu(U,V) e^{-\mathrm{str}\,[(\sigma-\rho)^\dagger(\sigma-\rho)]} = \frac{1}{(k!)^2} \frac{\det\gamma(s_p^0, r_q^0)\det\gamma(s_m^1, r_n^1)}{B(S^2)B(R^2)} \ , \qquad (4.179)$$

where $(s_p^0; is_p^1)$ and $(r_p^0; ir_p^1)$ (with $p = 1, \ldots, k$) denote the eigenvalues of σ and ρ, respectively. The quantity $\gamma(s,r)$ is defined as

$$\gamma(s,r) = \exp\left[-s^2 - r^2\right] I_0(2sr) \ . \qquad (4.180)$$

For a derivation of this integral we refer the reader to [103]. In this reference contributions from Efetov-Wegner terms [172] are discussed as well.

In (4.177), the matrix ρ is diagonal with diagonal elements R given by

$$R = \mathrm{diag}(x_1 + j_1, \ldots, x_k + j_k; x_1 - j_1, \ldots, x_k - j_k) \ , \qquad (4.181)$$

and the matrix σ is as given in (4.174). It is easy to verify that

$$\frac{1}{B(R^2)} = \prod_{p=1}^{k} 4x_p j_p \times (1 + O(j)) \, , \qquad (4.182)$$

which enables us to carry out the differentiations with respect to the source terms. We observe that, as a result of (4.176), the integrand of the generating function (4.177) factorizes into products and determinants of $k \times k$ matrices. By renaming the integration variables s_p^ϵ it can be shown that all $k!$ terms in the expansion of the determinants of $\gamma(s_p^0, r_q^0)$ and $\gamma(s_m^1, r_n^1)$ are equal. The only remaining determinant is the Berezinian (4.176). The products can then be absorbed into the determinant, which leads to the following determinantal structure for the correlation functions

$$\hat{R}_k(x_1, \ldots, x_k) \ = \ \det\left[\Sigma \hat{K}_N(x_p\Sigma, x_q\Sigma)\right]_{p,q=1,\ldots,k} \ , \qquad (4.183)$$

where the dimensionless kernel $\hat{K}_N(\xi,\eta)$ is given by

$$\begin{aligned} \hat{K}_N(\xi,\eta) \ &= \ -\frac{8N^2}{\pi}\sqrt{\xi\eta} \ \int_0^\infty dr \int_0^\infty ds \frac{rs}{r^2+s^2} \left(\frac{s^2+t^2}{-r^2+t^2}\right)^N \\ &\times \ e^{-N(r^2+s^2+(\xi^2-\eta^2)/2)} I_0\left(2Nr\xi\right) I_0\left(2Nis\eta\right) \ . \end{aligned} \qquad (4.184)$$

In this equation, we have introduced a dimensionless temperature

$$t = \pi T \Sigma \ . \qquad (4.185)$$

We have shown that all correlation functions follow from a single two-point kernel. This is the main virtue of the application of Guhr's supersymmetric method.

The spectral correlation functions $\hat{R}(x_1, \cdots, x_k)$ follow from the discontinuities of x_l across the real axis. The imaginary part in (4.184) arises as a result of the $-i\epsilon$ term in the s_p^0. One can easily convince oneself that the spectral correlators preserve the determinantal structure (4.183). We thus find

$$R_k(x_1, \ldots, x_k) = \det \left[\Sigma K_N(\Sigma x_p, \Sigma x_q) \right]_{p,q=1,\ldots,k} , \tag{4.186}$$

where

$$\begin{aligned} K_N(\xi, \eta) &= -\frac{8N^2}{\pi} \sqrt{\xi\eta} \int_0^\infty dr \int_0^\infty ds \frac{rs}{r^2 + s^2} \left(s^2 + t^2 \right)^N \\ &\times \mathrm{Im} \left(\frac{1}{-(r - i\epsilon)^2 + t^2} \right)^N \\ &\times e^{-N(r^2+s^2+(\xi^2-\eta^2)/2)} I_0 \left(2Nr\xi \right) I_0 \left(2Nis\eta \right) . \end{aligned} \tag{4.187}$$

Both in (4.184) and (4.187) the (x, y)-dependence has been chosen so that the kernel is symmetric in x and y when $t = 0$.

Both at $t = 0$ (section 6) and $t \neq 0$ (section 7) the two integrals can be separated by the Feynman method,

$$\frac{1}{r^2 + s^2} = N \int_0^\infty d\alpha e^{-N\alpha(r^2+s^2)} . \tag{4.188}$$

Most of the results leading to (4.183) can be generalized immediately to an arbitrary number of flavors. The main modifications are that the fermionic blocks will be of size $k + N_f$ rather than k, and that the supermatrices $\mathbf{x}$ and $\mathbf{j}$ in (4.153) will have N_f additional zero blocks. However, to the best of our knowledge, it is not possible to write the Berezinian as a determinant as in (4.176), and therefore the determinantal structure of (4.183) is lost. This means that it is no longer possible to express the correlation functions in terms of a single kernel.

On the other hand, work on both Wigner-Dyson random matrix models [230] and chiral random matrix models with arbitrary unitary invariant potentials [159, 3] using the orthogonal polynomial method shows that the correlation functions still possess a determinantal form. This suggests that the present approach might be modified to arbitrary N_f as well.

4.6.5 Exact evaluation of correlation functions for $T = 0$

In this section, we will show that kernel (4.187) at $T = 0$ reduces to the usual Laguerre kernel obtained by means of the orthogonal polynomial method [86, 155, 196].

After the introduction of new integration variables by $u = r^2$ and $v = s^2$, the kernel can be written as

$$\begin{aligned} K_N(x,y) &= -\frac{2N^3}{\pi}\sqrt{xy}e^{-N(x^2-y^2)/2}\int_0^\infty d\alpha \mathcal{I}_u(x)\mathcal{I}_v(y) \ , \\ \mathcal{I}_u(x) &= \int_0^\infty du \,\mathrm{Im}\left(\frac{1}{-u+i\epsilon}\right)^N I_0\left(2Nx\sqrt{u}\right)e^{-N(1+\alpha)u} \ , \\ \mathcal{I}_v(x) &= \int_0^\infty dv v^N I_0\left(i2Ny\sqrt{v}\right)e^{-N(1+\alpha)v} \ . \end{aligned} \tag{4.189}$$

In order to evaluate $\mathcal{I}_u$, we use the identity

$$\mathrm{Im}\,(u-i\epsilon)^{-N} = \pi\frac{(-1)^{N-1}}{(N-1)!}\frac{\partial^{N-1}}{\partial u^{N-1}}\delta(u) \tag{4.190}$$

in (4.189) and apply partial integrations in order to eliminate the derivatives of the delta function. As a result, we find

$$\mathcal{I}_u(x) = \frac{\pi(-1)^N}{(N-1)!}\frac{\partial^{N-1}}{\partial u^{N-1}}\bigg|_{u=0}\left[I_0\left(2Nx\sqrt{u}\right)e^{-N(1+\alpha)u}\right] \ , \tag{4.191}$$

which precisely describes the derivatives of the generating function for the Laguerre polynomials. (See eq. (4.208).) We thus obtain

$$\mathcal{I}_u(x) = -\frac{\pi(N(1+\alpha))^{N-1}}{(N-1)!}L_{N-1}\left(\frac{Nx^2}{1+\alpha}\right) \ , \tag{4.192}$$

where L_N are the Laguerre polynomials. The integral $_{\mathcal{I}v}$ is well-defined and can be found in the literature. (See eq. (4.209).) The result is

$$\mathcal{I}_v(x) = N!\frac{e^{-\frac{Ny^2}{(1+\alpha)}}}{(N(1+\alpha))^{N+1}}L_N\left(\frac{Ny^2}{1+\alpha}\right) \ . \tag{4.193}$$

After making a change of variables to $z = (1+\alpha)^{-1}$, we find for the two-point kernel that

$$K_N(x,y) = 2N^2\sqrt{xy}e^{-N(x^2-y^2)/2}\int_0^1 dz e^{-Ny^2z}L_{N-1}(Nx^2z)L_N(Ny^2z) \tag{4.194}$$

To further simplify (4.194), we will need the identity (see eq. (4.212))

$$e^{-z\eta}L_n(z\eta)L_{n-1}(z\xi) = \frac{d}{dz}\frac{e^{-z\eta}\left(L_{n-1}(z\xi)L_n(z\eta) - L_n(z\xi)L_{n-1}(z\eta)\right)}{\eta-\xi} \quad . \tag{4.195}$$

The integrand is now a total derivative, and we reproduce the well-known result [86, 155, 196]

$$\begin{aligned} K_N(x,y) &= 2N\frac{\sqrt{xy}}{x^2-y^2}e^{-N(x^2+y^2)/2} \\ &\times \left[L_{N-1}(Nx^2)L_N(Ny^2) - L_N(Nx^2)L_{N-1}(Ny^2)\right] \quad . \end{aligned} \tag{4.196}$$

This justifies our claim that the spectral correlation functions can be obtained by using a compact parametrization for the σ-variables.

It can be shown from the asymptotic properties of the Laguerre polynomials that spectral correlations in the bulk of the spectrum are given by the Gaussian unitary ensemble. The result for the microscopic region, $\tilde{x} = Nx \approx \mathcal{O}(1)$, follows from the asymptotic form of the Laguerre polynomials,

$$\lim_{n\to\infty} L_n(\frac{x}{n}) = J_0(2\sqrt{x}) \ , \tag{4.197}$$

which can be used after rewriting (4.196) with the aid of recursion relations for the Laguerre polynomials. As a result, we find the microscopic kernel

$$\begin{aligned} K_S(\tilde{x},\tilde{y}) &= \lim_{N\to\infty}\frac{1}{2N}K_N(\frac{\tilde{x}}{N},\frac{\tilde{y}}{N}) \\ &= \frac{\sqrt{\tilde{x}\tilde{y}}}{\tilde{x}^2-\tilde{y}^2}(\tilde{x}J_0(2\tilde{y})J_1(2\tilde{x}) - \tilde{y}J_0(2\tilde{x})J_1(2\tilde{y})) \ , \end{aligned} \tag{4.198}$$

which agrees with results obtained previously [181, 153, 154, 217, 17].

Finally, the microscopic spectral density is given by

$$\rho_S(\tilde{x}) = \lim_{y\to x} K_S(\tilde{x},\tilde{y}) \tag{4.199}$$

$$= \tilde{x}(J_0^2(2\tilde{x}) + J_1^2(2\tilde{x})) \ , \tag{4.200}$$

which is also in complete agreement with previous results.

4.6.6 Correlation functions at nonzero temperature

In this section we evaluate the microscopic limit of the imaginary part of the two-point kernel at nonzero temperature and show that, up to a scale

factor, it is in agreement with the zero temperature result in the limit $N \to \infty$. To this end, we perform the u and v integrals by a saddle point approximation. However, the saddle-point approximation to (4.187) suffers from the difficulty that $\bar{u} = -1 + t^2$ and $\bar{v} = 1 - t^2$. As a result, the pre-exponential factor diverges at the saddle-point. The aim of the transformations performed in the first part of this section is to eliminate this factor.

We begin by separating the r and s integrals (4.184) according to the Feynman method (4.188). The Feynman parameter, α, is replaced by the new integration variable $\beta = 1/\sqrt{(1+\alpha)}$. After rescaling $r \to \beta r$ and $s \to \beta s$, we find

$$\begin{aligned} \hat{K}_N(x,y) &= -\frac{16N^2}{\pi}\sqrt{\tilde{x}\tilde{y}}\int_0^1 \beta d\beta \int_0^\infty r dr \int_0^\infty s ds \left(\frac{s^2+t^2/\beta^2}{-(r-i\epsilon)^2+t^2/\beta^2}\right)^N \\ &\times \; e^{-N(r^2+s^2)+(\tilde{x}^2-\tilde{y}^2)/2} J_0\left(2i\beta r\tilde{x}\right) J_0\left(2\beta s\tilde{y}\right) , \end{aligned} \tag{4.201}$$

where we have also written the modified Bessel functions in terms of ordinary Bessel functions. Next, we express the product of the Bessel functions as a derivative of the microscopic kernel (4.198) according to the following remarkable identity (see eq. (4.214))

$$2\beta\sqrt{xy}J_0(2\beta x)J_0(2\beta y) = \frac{d}{d\beta}\beta K_S(\beta x, \beta y) \; . \tag{4.202}$$

This identity can be derived from eq. (4.195) using the asymptotic limit (4.197) of the Laguerre polynomials. After insertion of this identity in (4.201) and partial integration with respect to β, we find

$$\begin{aligned} \hat{K}_N(x,y) &= -\frac{8N^2}{\pi}\int_0^\infty dr \int_0^\infty ds \sqrt{rs} e^{-N(r^2+s^2)+(\tilde{x}^2-\tilde{y}^2)/2} \\ &\times \; \left\{ \left(\frac{s^2+t^2}{-(r-i\epsilon)^2+t^2}\right)^N K_S(ir\tilde{x}, s\tilde{y}) \right. \\ &\left. - \int_0^1 d\beta K_S(i\beta r\tilde{x}, \beta s\tilde{y}) \frac{d}{d\beta}\left(\frac{s^2+t^2/\beta^2}{-(r-i\epsilon)^2+t^2/\beta^2}\right)^N \right\} \; . \end{aligned} \tag{4.203}$$

The second term in this equation can be simplified further. We differentiate with respect to s and undo the change of integration variables at the beginning of this section, i.e., $r \to r/\beta$, $s \to s/\beta$, and $\alpha = (1-\beta^2)/\beta^2$. Finally, we

perform the integration with respect to α and obtain cancellation of the factor $r^2 + s^2$ which results from the differentiation. We find that the integrands of the two terms in (4.203) differ only by a factor $-t^2/(s^2+t^2)(-r^2+t^2)$. Thus, this equation can be rewritten as

$$\begin{aligned} \hat{K}_N(x,y) &= -\frac{8N^2}{\pi}\int_0^\infty dr \int_0^\infty ds \sqrt{rs} e^{-N(r^2+s^2)+(\tilde{x}^2-\tilde{y}^2)/2} \\ &\times \left(\frac{s^2+t^2}{-(r-i\epsilon)^2+t^2}\right)^N \left(1 - \frac{t^2}{(s^2+t^2)(-r^2+t^2)}\right) K_S(ir\tilde{x}, s\tilde{y}) \ . \end{aligned} \tag{4.204}$$

The saddle point evaluation of $\hat{K}$ should be performed separately in the microscopic limit and in the bulk of the spectrum. In the latter case, the asymptotic forms of Bessel functions enter in the saddle point equations. However, this is not the case in the microscopic limit with $\tilde{x}$ fixed in the thermodynamic limit. The saddle-point approximation in the microscopic limit is particularly simple. At the saddle point, we find

$$\begin{aligned} \bar{r}^2 &= -1 + t^2 \ , \\ \bar{s}^2 &= 1 - t^2 \ , \end{aligned} \tag{4.205}$$

and both the second derivatives with respect to r and s are equal to $4N(1-t^2)$. Note that the $\bar{r}^2$ is outside the integration domain. As discussed in section 4, the imaginary part of the integrand of $K_N(x,y)$ is an even function of r, which allows us to extend the integration range from $-\infty$ to ∞ at the cost of a factor of $1/2$. It is then clear that the integration path can be deformed to reach the saddle point $-i\sqrt{1-t^2}$. We cannot extend the integration path for the evaluation of the real part of $K_N(x,y)$, and it is not clear how the integrals can be performed by a saddle-point method.

As a result, we find

$$\lim_{N\to\infty} \frac{1}{2N} \hat{K}_N(\frac{\tilde{x}}{N}, \frac{\tilde{y}}{N}) = i\zeta K_S(\zeta\tilde{x}, \zeta\tilde{y}) \ . \tag{4.206}$$

For convenience, we have introduced the scaling factor $\zeta = \sqrt{1-t^2}$, which gives the temperature dependence of the spectral density in the neighborhood of $\lambda = 0$. We have shown that, up to this rescaling factor, the kernel K_N is independent of the temperature. This constitutes the central result of this paper.

4.6.7 Conclusions

In this paper, we have shown that *all* correlations of the eigenvalues near zero, measured in units of the average spacing, are independent of temperature deformations of the unitary chiral random matrix model. This result extends previous work on the microscopic spectral density. Together with other recent work on the universality of correlation functions with respect to deformations that preserve unitary invariance, this firmly establishes the universality of the complete eigenvalue distribution in the neighborhood of $\lambda = 0$.

It is our conjecture that the correlations of lattice QCD Dirac eigenvalues near zero virtuality are in the universality class of the chiral Gaussian Unitary Ensemble (chGUE). This conjecture has been supported by lattice simulations of the average microscopic spectral density via the valence quark mass dependence of the chiral condensate. In view of the present results, it would be interesting to study the correlations of lattice QCD Dirac eigenvalues in the neighborhood of $\lambda = 0$. Our prediction is that such correlations are given by the chGUE.

The present results were obtained by a generalization to the chGUE of the supersymmetric method developed by Guhr for the Gaussian Unitary Ensemble. The strength of this method is that it preserves the determinantal structure of the correlation functions. As usual in the supersymmetric formulation of random matrix theory, this method also requires a proper parametrization of the integration variables. Following work by Wegner and Efetov, it was believed that hyperbolic symmetry was an essential ingredient for the parametrization of the integration manifold. The surprising feature of the present method is that all *spectral* correlation functions can be obtained from a compact integration manifold. One reason might be that, because of $U_A(1)$ symmetry, the resolvent satisfies the relation

$$G(x + i\epsilon) = -G(x - i\epsilon), \tag{4.207}$$

and that all spectral correlation functions can therefore be obtained from a generating function that does not involve infinitesimal increments of opposite signs. However, this does not explain why Guhr's method also allows for a compact integration manifold in the case of the GUE. Clearly, more work is needed to address this issue.

Our results are based on the choice of a compact integration manifold. We have provided two important pieces of evidence supporting this choice. First, a detailed analysis of the imaginary part of the generating function of the one-point function shows that a non-compact integration domain can be transformed into a compact one. Second, a compact parametrization of the

integration manifold reproduces the exact correlation functions of the chGUE. However, the ultimate justification of this change of integration variables remains an open problem.

Appendix A: Some useful identities

A generating function for the generalized Laguerre polynomials is given by

$$\sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n+\alpha+1)} L_n^{\alpha}(x) = e^z \frac{1}{(xz)^{\alpha/2}} J_{\alpha}(2\sqrt{xz}) \ . \tag{4.208}$$

The Laguerre polynomials are defined by $L_n(x) = L_N^{\alpha=0}(x)$. A closely related integral is given by

$$\int_0^{\infty} dx x^{n+\alpha/2} e^{-ax} J_{\alpha}(2b\sqrt{x}) = \frac{n!}{a^{n+\alpha+1}} e^{-\frac{b^2}{a}} L_n^{\alpha}\left(\frac{b^2}{a}\right) \ . \tag{4.209}$$

The Laguerre polynomials satisfy the following remarkable identity

$$\begin{aligned} nL_{n-1}(z\eta)L_n(z\xi) &= (n-1)L_{n-1}(z\eta)L_{n-2}(z\xi) \\ &+ e^{z\eta}\frac{d}{dz}\left[ze^{-z\eta}L_{n-1}(z\eta)L_{n-1}(z\xi)\right] . \end{aligned} \tag{4.210}$$

By a recursive application of this relation and the Christoffel-Darboux formula,

$$\sum_{k=0}^{n-1} L_k(\xi)L_k(\eta) = \frac{n}{\eta-\xi}\left[L_{n-1}(\xi)L_n(\eta) - L_n(\xi)L_{n-1}(\eta)\right] \ , \tag{4.211}$$

we find

$$e^{-z\eta}L_n(z\eta)L_{n-1}(z\xi) = \frac{d}{dz}\frac{e^{-z\eta}\left(L_{n-1}(z\xi)L_n(z\eta) - L_n(z\xi)L_{n-1}(z\eta)\right)}{\eta-\xi} . \tag{4.212}$$

From the asymptotic from of the Laguerre polynomials,

$$\lim_{n\to\infty} n^{-\alpha} L_n^{\alpha}(\frac{x}{n}) = x^{-\alpha/2} J_{\alpha}(2\sqrt{x}) \ , \tag{4.213}$$

we obtain the following relation for Bessel functions

$$2\beta\sqrt{xy}J_0(2\beta x)J_0(2\beta y) = \frac{d}{d\beta}\beta K_S(\beta x, \beta y) \ , \tag{4.214}$$

where the Bessel kernel K_S is defined as

$$K_S(x,y) = \frac{\sqrt{xy}}{x^2-y^2}\left(xJ_0(2y)J_1(2x) - yJ_0(2x)J_1(2y)\right) \ . \tag{4.215}$$

This kernel can be obtained from the Laguerre kernel (4.196) with the help of the asymptotic result (4.214) after rewriting the Laguerre polynomials in the same order by means of the recursion relation

$$(\alpha+n)L_{n-1}^{\alpha} = xL_n^{\alpha+1} - (x-n)L_n^{\alpha} \ . \tag{4.216}$$

4.7 Universality in invariant deformations of general chiral ensembles

In the previous sections we have worked with non-invariant deformations of the probability density in chiral unitary ensembles and showed the universality of local spectral properties in the hard edge of the spectrum. This was done using the supersymmetry method where we have adopted a gaussian probability density for convenience.

In this section we will outline a proof of the universality of local spectral properties of chiral ensembles under non-gaussian but invariant deformations of the probability density in (4.3). The original proof has been given by the Copenhagen group [3, 160] and it employs the orthogonal polynomial method introduced in section 2.3 and the appendix. This proof is also of importance to the work outlined in the next section, where we relate universality statements for unitary ensembles to corresponding orthogonal and symplectic ensembles.

In the framework of the orthogonal polynomial method, all spectral quantities are expressed in terms of the orthogonal polynomials corresponding to the probability density of the ensemble. The correlator kernel (2.25), which is fundamental to almost all local spectral expressions can be analytically summed using the Christoffel-Darboux formula (A.6). Later the large N limit of local correlators (N being the size of the matrix) can be expressed using the asymptotic forms of the relevant orthogonal polynomials.

The fundamental observation of the Copenhagen group [3, 160] has been to realize that all orthogonal polynomials arising in chiral unitary ensembles with arbitrary invariant probability densities, have the same asymptotic form given in terms Bessel functions. In other words, they have generalized the following relation for Laguerre polynomials corresponding to the potential $x^a e^{-x}$

$$\lim_{k\to\infty}\left(\frac{1}{k^a}L_k^a\left(\frac{x}{k}\right)\right) = x^{-\frac{1}{2}a}J_a(2\sqrt{(}x)), \tag{4.217}$$

to be valid (except some scaling factors) for polynomials orthogonal with respect to any potential $x^a e^{-V(x)}$ for a polynomial $V(x)$ over the interval $[0, \infty)$. This then is used to prove the universality of the Bessel Kernel in invariant chiral unitary ensembles.

Central to the work of Akemann et al. [3, 160] is the following theorem:

Theorem *Let $\{p_n^a(x)\}$ be a set of orthogonal polynomials with respect to the measure*

$$dx x^a e^{-V(x)}, \quad V(x) = \sum_{k \geq 1} \frac{g_k}{k} x^k, a = 1, 2, \ldots$$

over the range $[0, \infty)$ whose moments are all finite. If the polynomials can be normalized according to $p_k^a(0) = 1$, then, for fixed $\xi = N^2 x$ and $t = k/N$, the following limiting relation holds:

$$\lim_{N \to \infty} p_k^a(\frac{\xi}{N^2})\Big|_{k=Nt} = a! \frac{J_a\left(u(t)\sqrt{\xi}\right)}{\left(u(t)\sqrt{\xi}/2\right)}.$$

where $J_a(x)$ are the Bessel functions and $u(t)$ depends on the particular form of the weight function via the following relations

$$u(t) = \int_0^t \frac{dt'}{\sqrt{r(t')}}, \quad t = \sum_i \frac{g_i}{2} \left(\begin{array}{c} 2i \\ i \end{array} \right) r(t)^i.$$

Here $r(t)$ is given by the asymptotic form of the coefficients of the three term recurrence relation (A.3). The assumption for nonvanishing $p_k^a(x)$ corresponds to a nonvanishing spectral density at the origin. As it is well known, this condition is necessary for the presence of the 'hard edge' of the spectrum. The Bessel kernel is no longer present when the microscopic spectral density vanishes. This point was also seen earlier in sections 4.3 and 4.5 where we have shown that a phase transition occurs to the local spectral properties at the point where the microscopic spectral density vanishes.

The proof of the theorem above for arbitrary a is given by induction on a where the case for $a = 0$ proceeds by establishing relations between the coefficients of the three term recurrence relation (A.3) by using the identities

$$\begin{aligned} 1 &= -\int_0^\infty dx \frac{d}{dx} \left[e^{-V(x)} p_k(x) p_k(x) \right] \\ &= \int_0^\infty dx V'(x) e^{-V(x)} p_k(x) p_k(x), \end{aligned} \tag{4.218}$$

$$
\begin{aligned}
0 &= -\int_0^\infty dx \frac{d}{dx}\left[xe^{-V(x)}p_k(x)p_k(x)\right] \\
&= \int_0^\infty dx x V'(x) e^{-V(x)} p_k(x)p_k(x) - h_k - 2h_k, \qquad (4.219)
\end{aligned}
$$

where h_k are normalization constants defined in the appendix.

The relations between recursion relation coefficients thus obtained are then used in their asymptotic form to establish a differential equation for $p_k^0(x)$ for large k, where the unique solutions are the aforementioned Bessel functions. The details are involved, for which refer the reader to the original references given above.

4.8 Relations between orthogonal, symplectic and unitary ensembles

The purpose of the work in this section is to relate the spectral information about the chiral unitary ensembles to corresponding chiral orthogonal and symplectic ensembles.

Orthogonal and symplectic ensembles are generally harder to study than the unitary ones. For example in the case of invariant ensembles the standard orthogonal polynomial method does not work for them, instead one has to introduce skew-orthogonal polynomials [67, 143] for the iterative reduction of the eigenvalue partition functions. Additionally, the Itzykson-Zuber integrals which proved so useful for unitary groups (and ensembles) are not known for orthogonal and symplectic ensembles. This has to do with the fact that the corresponding heat equations, as discussed in section 4.4 and 4.5, are not separable.

Because of these difficulties, studies of orthogonal and symplectic ensembles are not as abundant as the unitary ones. Specifically universality proofs for th em are much rarer to find. In this paper we try to remedy that situation by establishing a relation between the kernels of the three ensembles. The kernels carry all important technical information about the spectra. More precisely, the 'prekernels' of orthogonal and symplectic chiral ensembles are related to the kernel of the corresponding unitary ensemble by a differential and an integral operator, respectively. The motivation of this work is to extend the universality theorems known for unitary ensembles to similar orthogonal and symplectic ensembles. This is not yet a closed chapter and more work will be the subject of further studies.

This work was published in *Phys. Rev. Lett.* **81**: 248-251, (1998), which we reproduce except for minor changes.

The success of RMT is based on universality, and it is no surprise that it has received a great deal of attention in recent literature [12, 11, 10, 25, 47, 106, 121, 118, 48, 42] [231, 44, 64, 104, 125, 126, 39, 135, 136, 88, 80, 40] [26, 15, 7, 3, 4, 60, 227, 53]. What has been shown is that spectral correlators on the scale of the average eigenvalue spacing are insensitive to the details of the probability distribution of the matrix elements. Because of its mathematical simplicity most studies were performed for complex ($\beta = 2$) Hermitean RMT's. However, in the case of the classical RMT's it was shown that universality extends to real ($\beta = 1$) and quaternion real ($\beta = 4$) matrix ensembles [25, 106, 121]. This suggests that relations between correlation functions for different values of β which can be derived for a Gaussian probability distribution [203, 30, 28] might be valid for a wide class of probability distributions. The main goal of this paper is to establish such general relations. As a consequence, universality for the much simpler complex ensembles implies universality for the real and quaternion real ensembles.

In this letter we address the question of microscopic universality for the chiral ensembles. These ensembles are relevant for the description of spectral correlations of the QCD Dirac operator. They also appear in theory of universal conductance fluctuations in mesoscopic systems [181, 9]. In particular, they can be applied to spectral correlations near $\lambda = 0$. According to the Banks-Casher formula [22], this part of the spectrum is directly related to the order parameter Σ of the chiral phase transition ($\Sigma = \lim \pi\rho(0)/V$, where V is the volume of space time and $\rho(\lambda) = \sum_k \delta(\lambda - \lambda_k)$). It is therefore natural to introduce the microscopic limit where the variable $u = \lambda V\Sigma$ is kept fixed for $V \to \infty$. For example, the microscopic spectral density is defined by [180, 204]

$$\rho_S(u) = \lim_{V\to\infty} \frac{1}{V\Sigma}\langle\rho(\frac{u}{V\Sigma})\rangle, \tag{4.220}$$

where the average is over the distribution of the matrix elements of the Dirac operator. Successful applications of the chiral ensembles to lattice QCD spectra can be found in [30, 28, 142, 145, 113].

The chiral random matrix ensembles for N_f massless quarks in the sector of topological charge ν are defined by the partition function [180]

$$Z^{\beta}_{N_f,\nu} = \int DW \det{}^{N_f} \begin{pmatrix} 0 & iW \\ iW^\dagger & 0 \end{pmatrix} e^{-n\beta \mathrm{Tr} V(W^\dagger W)}, \tag{4.221}$$

where W is a $2n\times(2n+\nu)$ matrix. As is the case in QCD, we assume that ν does not exceed $\sqrt{2n}$. The parameter $2n$ is identified as the dimensionless volume of space time. The matrix elements of W are either real ($\beta = 1$, chiral Orthogonal

Ensemble (chOE)), complex ($\beta = 2$, chiral Unitary Ensemble (chUE)), or quaternion real ($\beta = 4$, chiral Symplectic Ensemble (chSE)). For technical reasons we only consider finite polynomials potentials $V(x)$. The simplest case is the Gaussian case with $V(x) = \Sigma^2 x$ (also known as the Laguerre ensemble).

It was shown by Akemann et al. [3] that, for $\beta = 2$, the microscopic spectral density and the microscopic spectral correlators do not depend on the potential $V(x)$ and is given by the result [217] for the Laguerre ensemble. For $\beta = 2$ all spectral correlators can be obtained from an orthogonal polynomial kernel corresponding to the probability distribution. In their proof the Christoffel-Darboux formula is used to express this kernel in terms of large order polynomials. Microscopic universality then follows from the asymptotics of orthogonal polynomials. As a remarkable achievement, they were able to generalize the relation for Laguerre polynomials

$$\lim_{n\to\infty} n^{-a} L_n^a(\frac{x}{n}) = x^{-a/2} J_a(2\sqrt{x}) \tag{4.222}$$

to orthogonal polynomials corresponding to an arbitrary polynomial potential. However, their work cannot easily be generalized to $\beta = 1$ and $\beta = 4$. The main result of the present work is a relation between the kernels for the correlation functions of the chOE and chSE and the kernel of the chUE. This relation is exact for the Gaussian ensembles and is valid asymptotically for an arbitrary polynomial potential. For $\beta = 4$, this relation shows universality of the microscopic spectral density and correlators (for $\beta = 1$ only a partial proof was obtained).

The partition function (4.221) is invariant under $W \to U^\dagger W V$ where the matrices U and V with dimensions determined by W are orthogonal for $\beta = 1$, unitary for $\beta = 2$, and symplectic for $\beta = 4$. This invariance makes it possible to express the partition function (4.221) in terms of the eigenvalues x_k of $WW^\dagger$ as

$$Z^\beta_{N_f,\nu} = \int \prod_k dx_k x_k^{2a} |\Delta(x_i)|^\beta e^{-n\beta \sum_k V(x_k)}, \tag{4.223}$$

where the Vandermonde determinant is defined by $\Delta(x_i) = \prod_{k<l}(x_k - x_l)$ and $a = N_f - 1 + \beta\nu/2 + \beta/2$.

For $\beta = 2$, the spectral correlation functions can be evaluated [148, 143, 146] by expressing the Vandermonde determinant in terms of the orthogonal polynomials defined by

$$\int_0^\infty dx e^{-2\phi_a(x)} q_k^{2a}(x) q_l^{2a}(x) = \delta_{kl}, \tag{4.224}$$

where we have introduced the potential $\phi_a(x) = nV(x) - a\log x$. By using orthogonality relations it can be shown that all spectral correlation functions can be expressed in terms of the kernel

$$K_{2n}^{2a}(x,y) = \sum_{k=0}^{2n-1} q_k^{2a}(x) q_k^{2a}(y). \tag{4.225}$$

The spectral density is given by $K_{2n}^{2a}(x,x)\exp(-2\phi_a(x))$. Microscopic universality then follows from the following generalization of (4.222) [3]

$$\lim_{n\to\infty} \sqrt{h_k^{2a}} q_k^{2a}(\frac{x^2}{n^2}, n)\Big|_{k=tn} = \Gamma(2a+1)\frac{J_{2a}(u(t)x)}{(u(t)x/2)^{2a}}, \tag{4.226}$$

in the normalization $q_k^{2a}(0)\sqrt{h_k^{2a}} = 1$. The function $u(t)$ follows from the asymptotic properties of the leading order coefficients of the $q_k^{2a}(x)$ and the normalizations h_k^{2a}. Its value at $t = 1$ is given by $u(1) = 2\pi\rho(0)$.

In order to perform the integrations by means of orthogonality relations for $\beta = 1$ and $\beta = 4$, one has to introduce the skew-orthogonal polynomials [67, 148, 143, 146]. Below, we first discuss the case $\beta = 1$ and then give general outlines for the case $\beta = 4$.

For $\beta = 1$, the skew orthogonal polynomials of the second kind are defined by

$$< R_i, R_j >_R = J_{ij}. \tag{4.227}$$

with the skew orthogonal scalar product

$$< f, g >_R = \int_0^\infty dx e^{-2\phi_a(x)} f(x)\hat{Z}g(x), \tag{4.228}$$

and nonzero matrix elements of J_{ij} given by $J_{2k,2k+1} = -J_{2k+1,2k} = -1$. The operator $\hat{Z}$ is defined by

$$\hat{Z}g(x) = \int_0^\infty dy e^{\phi_a(x)}\epsilon(x-y)e^{-\phi_a(y)}g(y), \tag{4.229}$$

Here, $\epsilon(x) = x/2|x|$. It can be shown that all correlation functions can be expressed in terms of the kernel [67, 148, 143, 146]

$$K_1(x,y) = \int_0^x dz e^{-\phi_a(z)} k_1(z,y) e^{-\phi_a(y)}, \tag{4.230}$$

where we have introduced the pre-kernel

$$k_1(y,z) = \sum_{i,j=0}^{2n-1} R_i(y) J_{ij} R_j(z). \tag{4.231}$$

In particular, the spectral density is given by

$$\rho(x) = K_1(x,x) - \frac{1}{2}K_1(\infty, x). \tag{4.232}$$

A general scheme for the construction of skew-orthogonal polynomials was introduced by Brézin and Neuberger[43]. The idea is to express them in terms of orthogonal polynomials defined by (4.224). For technical reasons we expand in the polynomials q_k^{2a+1} (with weight function $x^{2a+1}\exp(-2nV(x))$). The skew-orthogonal polynomials of degree i can thus be expressed as

$$R_i(x) = \sum_{j=0}^{i} T_{ij} q_j^{2a+1}(x), \tag{4.233}$$

where T is a lower triangular matrix with nonvanishing diagonal elements. An essential role is played by the inverse, $\hat{L}$, of the operator $\hat{X}^{-1}\hat{Z}$ with $\hat{Z}$ defined in (4.229) and $\hat{X}g(x) = xg(x)$. It can be easily verified that

$$\hat{L} = \hat{X}(\hat{\partial} - \phi_a'(\hat{X})) + \hat{1}. \tag{4.234}$$

The matrix representations of the operators $\hat{X}$, $\hat{X}\hat{\partial}$, $\hat{X}^{-1}\hat{Z}$ and $\hat{L}$ in the basis q_k^{2a+1} will be denoted by X_{kl}, D_{kl}, Y_{kl} and L_{kl}, respectively. In the remainder of this derivation the index $2a+1$ will be suppressed.

In matrix notation (4.227) can be rewritten as

$$TYT^T = -J. \tag{4.235}$$

By using that $LY = 1$, this relation can be expressed as

$$L = T^T J T. \tag{4.236}$$

It can be shown that the matrix L_{kl} is a band matrix with width determined by the order of the polynomial potential $V(x)$. It then follows that T is a band matrix as well [43]. For example, for a Gaussian potential we have that $T_{2m,k} = a_0\delta_{2m,k}$ and $T_{2m+1,k} = b_0\delta_{2m+1,k} + b_1\delta_{2m,k} + b_2\delta_{2m-1,k}$ with coefficients derived in [155, 203].

It turns out that we do not need explicit expressions for the T_{ij}. The pre-kernel (4.231) can be expressed as

$$k_1(x,y) = \sum_{i,j=0}^{2n-1} \sum_{k\le i} \sum_{l\le j} q_k(x) T_{ki}^T J_{ij} T_{jl} q_l(y). \tag{4.237}$$

In this relation the indices i and j run up to $2n-1$ in contradistinction to the relations (4.236) where they run up to ∞. However, it follows from the

band structure of L that the number of terms outside the range in (4.237) is of the same order as the degree of the polynomial potential which is finite. These terms are negligible in the continuum limit of the type (4.226) where the $q_k^{2a+1}(x)$ and L_{kl} depend smoothly on k and l (notice that L_{kl} is not smooth in $|k-l|$). However, for x around zero and y near the largest zero of $q_l(y)$ we expect potentially non-negligible contributions. We thus have that

$$k_1(x,y) \simeq \sum_{k,l=0}^{2n-1} q_k(x) L_{kl} q_l(y). \tag{4.238}$$

By means of a partial integration the matrix elements of $\hat{1} - \hat{X}\phi_a'(\hat{X})$ in L can be expressed in terms of the matrix elements of $\hat{X}\hat{\partial}$. This results in

$$\begin{aligned} L_{kl} &= \frac{1}{2}\int_0^\infty z dz e^{-2\phi_a(z)} (q_l(z) z \partial q_k(z) - q_k(z) z \partial q_l(z)) \\ &= \frac{1}{2}(D_{kl} - D_{lk}). \end{aligned} \tag{4.239}$$

The matrix elements of D can be re-expressed as $x\partial_x$ or $y\partial_y$. We finally arrive at a remarkably simple expression for $k_1(x,y)$,

$$k_1(x,y) \simeq \frac{1}{2}(y\partial_y - x\partial_x) K_{2n}^{2a+1}(x,y). \tag{4.240}$$

With the help the asymptotic properties of the q_k^{2a} (which are the same as for the Laguerre polynomials) this relation can be further simplified to (up to an overall factor determined by the average spectral density)

$$k_1(x,y) \sim \frac{1}{2}(\partial_y - \partial_x) K_{2n}^{2a}(x,y). \tag{4.241}$$

This is the central result of this paper. It is valid asymptotically both at the hard and the soft edge of the spectrum where a continuum limit of the orthogonal polynomials q_k^{2a+1} exists. However, as will be argued below, the result (4.241) is not valid for x near the hard edge and y at the soft edge of the spectrum. This result relates the orthogonal pre-kernel to the unitary kernel $K_{2n}^{2a}(x,y)$ which has been studied elaborately in the literature [49, 86, 196]. The relation (4.241) is exact for a Gaussian potential in which case it coincides with the result obtained in [203, 152, 153, 154, 42, 9].

Universality of the unitary kernel $K_{2n}^{2a}(x,y)$ at the hard edge has been well established [3] for the chiral ensembles, whereas universality at the soft edge was shown in [39, 135, 136]. We therefore expect universal behavior of $k_1(x,y)$ in these domains.

Let us finally focus on the spectral density. Using (4.232) and (4.241), for a Gaussian potential it can be expressed as

$$\rho(x) \simeq e^{-\phi_a(x)} \int_0^\infty dy e^{-\phi_a(y)} \epsilon(x-y) \frac{1}{2}(\partial_y - \partial_x) K_{2n}^{2a}(x,y). \tag{4.242}$$

In the microscopic limit where $n \to \infty$ at fixed $z \equiv xn^2$ the factor $\exp(-2nV(x))$ $\to 1$ and K_{2n}^{2a} approaches its universal limit. However, in one of the terms contributing to the integral the microscopic limit and the integration cannot be interchanged. It can be shown that there is an additional contribution with x near zero and y near the edge of the spectrum. Naively taking into account this contribution for nongaussian potentials leads to a microscopic spectral density that differs from the universal expression. Alternatively, we have established universality of the microscopic spectral density by means of Monte-Carlo simulations. Apparently, the smoothness assumptions in the derivation of (4.240) are violated in this case. The edge contribution resides in the term $K_1(x,\infty)$ in (4.232). In the first term contributing to spectral density, $K_1(x,x)$, the microscopic limit and the integral can be interchanged. This establishes universality of $K_1(x,x)$.

In the case of a Gaussian potential the edge contribution can be obtained from the asymptotic expansion of the Laguerre polynomials in this region (an expression in terms of Airy functions). In a future publication, we hope to establish a possible relation with universal behavior of the q_k^{2a} near the edge of the spectrum [39, 135, 136].

The above analysis carries through for the symplectic ensemble. In this case there are no contributions from the soft edge and universality of the microscopic spectral density can be shown rigorously. For $\beta = 4$ (with an additional factor $1/2$ in the exponent of (4.223)), the correlation functions can be expressed in terms of the kernel

$$k_4(x,y) = \sum_{i,j=0}^{2n-1} Q_i(x) J_{ij} Q_j(y), \tag{4.243}$$

where the $Q_i(x)$ are skew orthogonal polynomials of the first kind which are defined by the skew-scalar product

$$< f, g >_Q = \int_0^\infty \frac{dx}{x} e^{-2\phi_a(x)} f(x)(\hat{L} - \hat{1}) g(x), \tag{4.244}$$

with the operator $\hat{L}$ defined in (4.234). In this case we express the $Q_i(x)$ in

terms of the polynomials $q_k^{2a-1}(x)$,

$$Q_k(x) = \sum_{l=0}^{k} S_{kl} q_l^{2a-1}(x). \tag{4.245}$$

The matrix elements of the operators are also in this basis. From the orthogonality relation $< Q_k, Q_l >_Q = J_{kl}$ it can be shown that $SLS^T = -J$ from which we derive $S^T J S = XYX^{-1}$. Again, due to the band structure of L_{kl}, the range of the summations in this relation and in (4.243) differs by a finite number of terms which can be neglected in the continuum limit. We thus find

$$\begin{aligned} k_4(x,y) &\simeq \sum_{k,l=0}^{2n-1} q_k^{2a-1}(x) \left(\hat{Z}\hat{X}^{-1}\right)_{kl} q_l^{2a-1}(y) \\ &= e^{\phi_a(y)} \int_0^\infty \frac{dz}{z} e^{-\phi_a(z)} \epsilon(y-z) K_{2n}^{2a-1}(x,z). \end{aligned} \tag{4.246}$$

Universality of $k_4(x,y)$ thus follows from universality of $K_{2n}^{2a-1}(x,z)$. This relation is exact for a Gaussian potential and reproduces the result found in [30, 28].

In conclusion, we have shown that relations between the kernels for the chOE and chSE and the kernel for chUE are not accidental but follow from an intriguing underlying mathematical structure. Under certain smoothness assumptions these relations are valid asymptotically for an arbitrary polynomial potential. Microscopic universality for $\beta = 4$ and in part for $\beta = 1$ thus follows from universality at $\beta = 2$ at hard edge of the spectrum.

Chapter 5

Conclusions and outlook

In this thesis we have contributed to establishing and extending the universality of the local spectral characteristics of the Dirac operator in quantum chromodynamics. We have also developed novel mathematical tools which can be used in similar studies. A central theme to this study has been the view of random matrix models as minimal information models, which can be utilized to extract universal properties of a 'complex' dynamical system.

It is the sincere hope of this author that this kind of an approach of extracting universal properties by the use minimal information models will become more widespread in the future, especially in cases where the dynamical systems in question are of social or biological origin. In those cases there is a much larger degree of complexity present. In the study of such 'complex systems' (as they are loosely called) one typically constructs very simplistic models (if modeling is attempted at all, which might be viewed to be marginal in some fields). However any such model involves ad hoc decisions in its construction and it is necessary to distinguish between model specific artifacts and universal features that even a simplistic model shares with the actual problem. We hope that the perspective outlined above will eventually be of some help in these types of problems as well. However this shall be the subject of other studies.

Appendix A

Orthogonal polynomials

Orthogonal polynomials occur frequently in various studies in theoretical physics. The orthogonal polynomial method outlined in chapter 2 is one of the strongest tools in the arsenal of random matrix theory - whenever it is applicable which is not always the case.

Below we outline basic definitions and theorems about orthogonal polynomials which are used throughout this study. More detailed accounts can be found in many standard references including [191, 23, 94].

Let an interval (a, b) on the real line and a nonnegative real weight function $w(x)$ defined on (a, b) be given where

$$\int_a^b dx\ w(x)x^k, \quad k = 0, 1, 2, \ldots \tag{A.1}$$

exist. Then there exist a set of polynomials $p_0(x), p_1(x), p_2(x), \ldots$, such that $p_k(x)$ is a kth order polynomial with a nonzero coefficient, a_k, for x^k and that this set of polynomials are orthogonal with respect to the inner product given by

$$\int_a^b dx\ w(x)p_k(x)p_l(x) = h_k\delta_{kl}. \tag{A.2}$$

This set of polynomials, $\{p_k(x)\}$, are called a system of orthogonal polynomials on (a, b) with the weight function $w(x)$.

Two of the most common normalization conventions are to either take the $\{p_k(x)\}$ to be orthonormal (i.e. $h_k = 1$ for all k) or to take them to be monic (i.e. the coefficient, a_k, of x^k in $p_k(x)$ is taken to be unity).

It can be shown that among any set orthogonal polynomials there is a *three term recurrence relation*:

$$p_k(x) = (A_k x + B_k)p_{k-1}(x) - C_k p_{k-2}(x), \quad k \geq 2, \tag{A.3}$$

where we have assumed that $h_k = 1$ and A_k, B_k and C_k are constants. Furthermore, we have

$$A_k = \frac{a_k}{a_{k-1}}, \tag{A.4}$$

$$C_k = \frac{a_k a_{k-2}}{a_{k-1}^2}. \tag{A.5}$$

The three term recurrence relation can be used recursively to prove an identity which expresses sums of polynomials of all orders in terms of high order polynomials: (we again assume $h_k = 1$.)

$$\sum_{l=0}^{k} p_l(x)p_l(y) = \frac{a_k}{a_{k+1}} \frac{p_{k+1}(x)p_k(y) - p_k(x)p_{k+1}(y)}{x - y}. \tag{A.6}$$

This relation is known as the *Christoffel-Darboux identity*. It is extremely important in invariant random matrix models as the kernel of correlation functions can be expressed in terms of the left hand side of this relation with corresponding orthogonal polynomials. Then using the right hand side of (A.6) and the asymptotic forms of the orthogonal polynomials, closed analytic forms for the correlation functions can be obtained.

Below we list the properties of two of the most utilized orthogonal polynomials in random matrix theory.

The *Hermite polynomials* occur frequently in the study of the gaussian unitary ensemble in random matrix theory. They correspond to the interval $(-\infty, +\infty)$ and the weight function $w(x) = e^{-x^2}$. A generating function for them can be written as

$$H_k(x) = (-1)^k e^{x^2} \frac{d^k}{dx^k}(e^{-x^2}). \tag{A.7}$$

The lowest order Hermite polynomials are

$$H_0(x) = 1, \tag{A.8}$$

$$H_1(x) = 2x, \tag{A.9}$$

$$H_2(x) = 4x^2 - 2, \tag{A.10}$$

$$H_3(x) = 8x^3 - 12x. \tag{A.11}$$

The study of the chiral gaussian unitary ensemble naturally results in the appearance of *Laguerre polynomials*. They correspond to the interval $(0, +\infty)$ and the weight function $w(x) = x^a e^{-x}$. This happens because the spectrum is

symmetric around zero with eigenvalues $\pm\lambda_k$ and it becomes natural to work with the squares of the eigenvalues $x_k = \lambda_k^2$.

The generating function for the Laguerre polynomials is given by the Rodrigues formula:

$$L_k^a(x) = \frac{1}{k!} e^x x^{-a} \frac{d^k}{dx^k} \left(e^{-x} x^{k+a} \right). \tag{A.12}$$

The lowest order Laguerre polynomials are given by

$$\begin{aligned}
L_0^a(x) &= 1, && \text{(A.13)}\\
L_1^a(x) &= -x + a + 1, && \text{(A.14)}\\
L_2^a(x) &= \frac{1}{2}x^2 - (a+2)x + \frac{1}{2}(a+1)(a+2), && \text{(A.15)}\\
L_3^a(x) &= -\frac{1}{6}x^3 + \frac{1}{2}(a+3)x^2 - \frac{1}{2}(a+2)(a+3)x \\
&\quad + \frac{1}{6}(a+1)(a+2)(a+3). && \text{(A.16)}
\end{aligned}$$

A remarkable identity relates Laguerre polynomials to Bessel functions which is used in the derivation of the Bessel kernel of the chiral gaussian unitary ensemble :

$$\lim_{k\to\infty} \left(\frac{1}{k^a} L_k^a \left(\frac{x}{k} \right) \right) = x^{-\frac{1}{2}a} J_a(2\sqrt{(}x)). \tag{A.17}$$

Appendix B

Superanalysis

In this appendix we outline the basics of superanalysis and fix our notation and conventions.

'Super'-mathematics, which was pioneered by Berezin's work [32], refers to the usage of commuting and anti-commuting degrees of freedom on an equal footing. Physically this became relevant with the invention of supersymmetric field theories where there exist a symmetry operation which converts fermionic fields to bosonic fields and vice versa [158]. The corresponding mathematical techniques have been successfully introduced into random matrix theory after Efetov's work [76].

Below we will proceed with 'super'-generalizations of algebra, linear spaces and operators, operations on linear operators such as transposition, definitions for differentiation and integration etc. A more complete treatment can be found in Berezin's book [32]. We mostly follow the conventions used in [216].

An anticommuting algebra, or a Grassmann algebra, with N generators, χ_k, is given by the following relations:

$$\chi_j \chi_k = -\chi_k \chi_j, \tag{B.1}$$

from which follows that all generators are nilpotent, $\chi_k^2 = 0$.

Conventionally one usually works with 'complex' Grassmann algebras with generators, χ_k and χ_k^*, where χ_k and χ_k^* are viewed as independent generators. We will adopt the conjugation of the second kind where

$$(\chi_k)^* = \chi_k^*, \tag{B.2}$$

and

$$(\chi_k^*)^* = -\chi_k. \tag{B.3}$$

Furthermore we have

$$(\chi_k \chi_l)^* = \chi_k^* \chi_l^*. \tag{B.4}$$

A (p, q)-supervector (or a graded vector as it is sometimes called) has p commuting components, z_k, and q anticommuting components, ζ_l:

$$\Phi = \begin{pmatrix} z \\ \zeta \end{pmatrix}. \tag{B.5}$$

A (p, q)-supermatrix (or a graded matrix) acts on the space of (p, q)-supervectors. Hence it has the form

$$F = \begin{pmatrix} a & \sigma \\ \rho & b \end{pmatrix}, \tag{B.6}$$

where a and b are $p \times p$ and $q \times q$ commuting matrices and σ and ρ are $p \times q$ and $q \times p$ anticommuting matrices, respectively.

Transposition on supermatrices are defined in such a way that

$$(F\Phi)^T = \Phi^T F^T. \tag{B.7}$$

Thus we have

$$F^T = \begin{pmatrix} a^T & \rho^T \\ -\sigma^T & b^T \end{pmatrix}. \tag{B.8}$$

Furthermore we define hermitian conjugation in the usual way

$$F^\dagger = \left(F^T\right)^*, \tag{B.9}$$

so that $(F\Phi)^\dagger = \Phi^\dagger F^\dagger$.

For two Grassmann vectors ξ and η we have

$$\left(\xi \eta^T\right)^T = -\eta \xi^T \tag{B.10}$$

and therefore for any supervector Φ

$$\left(\Phi \Phi^\dagger\right)^\dagger = \Phi \Phi^\dagger \tag{B.11}$$

and for any supermatrix F

$$\left(F^\dagger F\right)^\dagger = F^\dagger F \tag{B.12}$$

as in the case of ordinary matrices.

Supertrace and superdeterminants (again sometimes called graded trace and graded determinants) are introduced in such a way to satisfy the relation

$$\mathrm{sdet}F = \exp\left(\mathrm{str}\,\ln F\right). \tag{B.13}$$

Therefore we introduce

$$\mathrm{str}\,F \equiv \mathrm{tr}\,a - \mathrm{tr}\,b \tag{B.14}$$

and

$$\mathrm{sdet}F \equiv \det(a - \sigma b^{-1}\rho)(\det b)^{-1}. \tag{B.15}$$

These definitions satisfy all useful properties of ordinary traces and determinants. Namely,

$$\begin{aligned}
\mathrm{str}\,F^T &= \mathrm{str}\,F, && \text{(B.16)}\\
\mathrm{str}\,(FG) &= \mathrm{str}\,(GF), && \text{(B.17)}\\
\mathrm{str}\,(\Phi_1\Phi_2{}^\dagger) &= \phi_1{}^\dagger\cdot\phi_2, && \text{(B.18)}\\
\mathrm{sdet}F^T &= \mathrm{sdet}F, && \text{(B.19)}\\
\mathrm{sdet}(FG) &= \mathrm{sdet}(F)\mathrm{sdet}(G), && \text{(B.20)}
\end{aligned}$$

where $\phi_1{}^\dagger\cdot\phi_2$ denotes inner product of two supervectors generalized in the natural way.

Most of the useful framework in linear algebra admits a straightforward 'super'-generalization. For example if we introduce superunitary matrices as in

$$U^\dagger U = UU^\dagger = 1 \tag{B.21}$$

than it can be shown that any hermitian supermatrix, H, can be 'superdiagonalized' in the usual way

$$H = U^\dagger \Lambda U, \tag{B.22}$$

where

$$\Lambda = \begin{pmatrix} \lambda_b 0 \\ 0 \lambda_f \end{pmatrix}, \tag{B.23}$$

with λ_b and λ_f ordinary diagonal real matrices. The eigenvalues in λ_b are called bosonic whereas the ones in λ_f are called fermionic.

Even some results in harmonic analysis over unitary groups generalize rather naturally to superunitary groups as we discuss in chapter 4.

Differentiation of Grassmann numbers are introduced in the natural way

$$\frac{\partial}{\partial \chi_k}\chi_l = \delta_{kl}. \tag{B.24}$$

This implies, together with the nilpotency of Grassmann generators, that the Taylor expansion of any Grassmann valued function truncates after a finite number of terms. As it is seen in chapter 4 this provides a number of advantages in computations when the problem can be mapped into a supersymmetric formulation.

The integration of Grassmann numbers are introduced in such a way to satisfy the following identity

$$2\pi \int d\chi^* d\chi \exp\left(i\chi^*\chi\right) = 1, \tag{B.25}$$

where χ and χ^* are Grassmann numbers as before and the factor 2π is conventional. It is chosen in such a way as to satisfy (by cancelling the 'bosonic prefactor')

$$\int d\phi^* d\phi d\chi^* d\chi \exp\left(i\phi^*\phi + i\chi^*\chi\right) = 1, \tag{B.26}$$

where ϕ is an ordinary complex number.

We therefore introduce the following conventions for Grassmann integration

$$\int d\chi \;\; = \;\; 0 \tag{B.27}$$

$$\int d\chi \; \chi \;\; = \;\; \frac{1}{\sqrt{2\pi}} \tag{B.28}$$

As any Grassmann valued function can be expanded linearly in terms of Grassmann generators these two identities are sufficient for defining all integrations.

Another useful identity follows directly from these definitions: In the case ordinary complex vectors and determinants we have

$$\int d[s^*]d[s] \exp\left(is^\dagger H s\right) = \frac{1}{\det\left(H/2\pi\right)}, \tag{B.29}$$

where s is a complex vector and H is an ordinary hermitian matrix. Using a Grassmann vector ζ we arrive at a similar identity

$$\int d[\zeta^*]d[\zeta] \exp\left(i\zeta^\dagger H \zeta\right) = \det\left(H/2\pi\right). \tag{B.30}$$

Furthermore for a supervector, Φ, and a hermitian supermatrix, H, we have the identity

$$\int d[\Phi^*]d[\Phi]\exp\left(i\Phi^\dagger H\Phi\right) = \frac{1}{\mathrm{sdet}\,(H)}. \tag{B.31}$$

This identity will play a central role in some of the computations throughout this work.

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